Module 473
Kepler's Laws and the Inverse Square Law
A.M. Fink

Applications of Calculus to Physics
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KEPLER'S LAWS AND THE INVERSE SQUARE LAW

by

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Title: KEPLER'S LAWS AND THE INVERSE SQUARE LAW

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Prerequisite Skills:
1. Know the elementary facts about vector algebra and vector differentiation.
2. Know how to compute area and arclength in polar coordinates.
3. Know the key features of the conic sections.
4. Be conversant with elementary integration.
5. Have the ability to change units of measurements and know how to write numbers in scientific notation.

Output Skills:
1. Be able to handle motion problems in polar coordinates, especially those concerning gravitational interaction.
2. State the relationships between Kepler's Laws and the Inverse Square Law.
3. Discuss the history and thought going into scientific development of an idea and the interaction between experiment and theory, at least in the context of planetary motion.

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1. DESCRIPTION OF THE PROBLEM

The orbit of a satellite around the earth may be considered to be determined by the gravitational interaction between it and the earth alone. Since the sun also affects the motion of the satellite, a small amount of error is introduced. However, the earth's gravity is the most important gravitational factor.

In the same way, the description of the motion of a planet around the sun may be viewed as the result of these two bodies' mutual gravitational attraction. Again other planets affect this motion, but the sun is the major influence on the orbit of a planet. The dynamics of the history of the solution of the problem of describing the motion of a planet can be recreated in a short time using the modern conveniences of vector differentiation. The central ideas and facts are the Inverse Square Law and Kepler's three Laws.

The common thread of the satellite and planetary motions reappears in modern physics on the sub-molecular level in the form of Coulomb potentials.

2. STATEMENT OF THE LAWS

We imagine the situation of two point masses, one mass very much larger than the other. The effect of this assumption is that we let the position of the larger mass be fixed. Put the origin of the coordinate system at the larger mass M. For convenience, call it the Sun. The motion of the smaller mass m, now called a planet describes a curve in 3-space. In Section 6 we will show that the motions of interest are planar. We will describe the motions in polar coordinates. The path of the planet is written in parametric form \((r(t), \theta(t))\) where \(t\) is time.
Figure 1. A planet moving around the sun in a fixed coordinate system. The heliocentric view of the solar system provided by Aristarchus c. 310-230 B.C. and resurrected by Copernicus, 1473-1543 led to the discovery of Kepler's Laws and the Inverse Square Law.

Two of Newton's Laws are germane to our discussion. The first is

\[ F = mA \]  

that is, force F and acceleration A are vector quantities and they are proportional with the proportionality constant being the mass \( m \). In our problem, the acceleration that accounts for the vector motion \( (r, \theta) \) is due to the external forces via equation (1).

Secondly, we have the Inverse Square Law

\[ F = \frac{GmM}{r^2} U \]

where \( G \) is a constant which depends on the units of measurement, but not on the solar system, and \( U \) is a unit vector directed from the origin to the mass \( m \).

We will discuss three of Kepler's Laws.

**Kepler's First Law.** The radius vector to the planet sweeps out area at a constant rate with respect to time, that is, the area of the shaded region in Figure 2 is \( \lambda|t_2 - t_1| \) where \( \lambda \) is a constant.

---

*As for linear motion, acceleration is the rate of change of the velocity vector, i.e., \( \lambda(t) = dV/dt \); and the velocity vector is in turn given by \( V(t) = dR/dt \) where \( R(t) \) is the position vector of the mass \( m \).
Figure 2. The radius vector sweeping out area. The first of Kepler's Laws was the computation of the rate in which this area is swept out. It is a constant, even for comets that pass the sun only once.

Kepler's Second Law. The planet's orbit is an ellipse with the sun at a focus.

Kepler's Third Law. If a is the semi-major axis of an elliptical orbit and T the time to complete one orbit, then $T^2/a^3$ is a solar system constant. In other words, if the quantity $T^2/a^3$ is computed for two different planets of the solar system, it is the same in both cases. It depends only on the mass of the sun and units.

3. HISTORY OF THE PROBLEM

The problem of trying to explain the planetary motion goes back to antiquity. The part of the history which we describe begins in the sixteenth century with Copernicus (1473-1543) who was a proponent of the heliocentric view of astronomy, i.e. that the "sun is at the center of things."

Tycho Brahe (1546-1601) opposed the Copernican theory on religious grounds. It was thought that anything that
did not put the earth at the center degraded humanity. Nevertheless, this astronomer's careful work was a major contribution in validating the Copernican approach. He had received a commission from King Frederick II of Denmark to update astronomical tables. His observatory on the island of Hven contained no telescope (it was invented in 1609) but he was nevertheless able to record a great deal of accurate information.

This accurate information was put to good use by Johann Kepler (1571-1630), who was Tycho's assistant for a short time. Trained as a mathematician, he took as his task the study of the orbit of Mars. He was an ardent supporter of the Copernican theory and his lifelong ambition was to find the mystical harmony in the skies. His detailed study of Mars led to his publishing his first two laws in 1609 and the third some ten years later.

Galileo Galilei (1564-1642), who is well known for his experiments on particles moving under the influence of gravity, dismissed Kepler's astronomy because in introducing ellipses he was departing from the more perfect circular motion. Thus the scientist who was to be branded a heretic for his scientific views rejected, out of hand, the work of Kepler for very unscientific reasons.

In the year that Galileo died, another scientist, Sir Isaac Newton (1642-1727) was born. Newton was well acquainted with the work of both Kepler and Galileo and of course with the Copernican approach. At the age of 25 he discovered that the only gravitational force consistent with Kepler's laws was the Inverse Square Law. He did not publish his result immediately because he attempted to validate it by doing calculations on the orbit of the moon. Unfortunately they did not check because some of the data on distance to the moon was incorrect. The correct data indeed did verify the Inverse Square Law. He published his result only when it was begun to be proposed by other scientists.
An amusing sidelight is that when asked about the possibility of the Inverse Square Law as being correct, Newton replied that he had once done the calculations. When asked to reproduce them, he could not! Eventually he found an error in his second calculation which when corrected gave the correct answer.* He is generally credited with the discovery of the Inverse Square Law.

In succeeding sections we try to recreate the scientific process of going from Tycho's empirical data to Kepler's Laws to Newton's Laws. We will also show that starting with Newton's Laws, we can recover Kepler's Laws.

4. MOTIONS DESCRIBED IN POLAR COORDINATES

Suppose that we have a motion described in polar coordinates and \( R(t) \) is the position vector. In order to isolate certain aspects of the motion we can introduce a local coordinate system as follows.

![Diagram of polar coordinates](image)

Figure 3. The standard unit vectors for parametric polar coordinate motions. These form a "moving coordinate system" which depend on the position of the particle. This modern tool, not available to Kepler and Newton, allows one to almost completely dispense with geometric and/or trigonometric arguments.

*This is a warning to the scientific neophyte. Keep your notebooks orderly.
The vectors \( U_r \) and \( U_\theta \) are to be unit vectors oriented as indicated. They do not depend on \( r \) but it is easy to see that \( U_r(\theta) = (\cos \theta, \sin \theta) \) and \( U_\theta(\theta) = (-\sin \theta, \cos \theta) \), (draw the triangles whose hypotenuses are \( U_r \) and \( U_\theta \) respectively)

\[
\frac{d}{d\theta} U_r = U_\theta; \quad \text{and} \quad \frac{d}{d\theta} U_\theta = -U_r.
\]

Now \( R(t) = r(t)U_r(\theta(t)) \) is the equation of the motion in polar coordinates. We let \( V \) and \( A \) be the velocity and acceleration vectors, the prime notation means differentiation with respect to time. Then

\[
V = R' = r'U_r + \frac{dr}{d\theta} \frac{d\theta}{dt} = r'U_r + rU_\theta \theta'
\]

and

\[
A = V' = (r''U_r + r'U_\theta \theta') + (r'U_\theta \theta' + rU_\theta \theta'' - r(\theta')^2U_r).
\]

Therefore

\[
A = [r'' - r(\theta')^2]U_r + (2r' \theta' + r \theta'')U_\theta.
\]

The coefficients in this vector

\[
a_r = r'' - r(\theta')^2
\]

and

\[
a_\theta = 2r' \theta' + r \theta''
\]

are called the radial and angular components of acceleration respectively. These are the usual tangential and normal components only in special cases, e.g. when the motion is on a circle centered at the origin.

We recall also that in polar coordinates the area element is \( r\,dr\,d\theta \) so that the area of the shaded region in Figure 2 is given by

\[
S(t_2) = \int_{\theta_1}^{\theta_2} \int_0^{R(\theta)} r\,dr\,d\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\,d\theta = \int_{t_1}^{t_2} \frac{1}{2}r^2\,d\theta \, dt.
\]
5. DEDUCTION OF THE INVERSE SQUARE LAW 
FROM KEPLER'S LAWS

Kepler's Laws are empirical results based on careful observations. If one assumes that only forces external to the planet account for its motion, then what must this force be? Let us assume Kepler's Laws.

First observe that the first law applied to the formula (7) gives via the Fundamental theorem of Calculus that

\[ S'(t) = \frac{1}{2} r^2 \theta' = \frac{\lambda}{2} \]

where \( \lambda \) is a constant.

The constant is divided by 2 to make the next equation and its further uses simpler. Thus

(8) \[ r^2 \theta' = \lambda \]

If we differentiate this relation then

(9) \[ 0 = 2rr' \theta' + r^2 \theta'' = r[2r' \theta' + r \theta''] + ra_\theta \]

We see that Kepler's first law implies that \( a_\theta = 0 \), that is the acceleration and therefore the force is purely in the radial direction. Such forces are called central force fields.

We further assume that the planet moves in a conic section (see Appendix 1), that is

(10) \[ r(1 + e \cos (\theta + \alpha)) = B \]

Differentiating with respect to time we get

\[ r'(1 + e \cos (\theta + \alpha)) - re \sin (\theta + \alpha) \theta' = 0 \]

If we multiply this equation by \( r \) and use (8) and (10) we get

\[ Br' - le \sin (\theta + \alpha) = 0 \]

Now we differentiate again to get

(11) \[ Br'' - le \cos (\theta + \alpha) \theta' = 0 \]
We solve (10) for \( \cos(\theta + \alpha) \) and use in (11) to get

\[
Br'' - \lambda\left(\frac{\alpha}{r} - 1\right)\theta' = 0.
\]

Rearranging, we have

\[
t'' - \frac{\lambda}{r} \theta' = -\frac{\lambda}{B} \theta'.
\]

On the left hand side replace \( \lambda \) by (8) and on the right side replace \( \theta' = \lambda r^{-2} \) by (8) to get:

\[
a_T = t'' - t(\theta')^2 = -\frac{\lambda^2}{Br^2}.
\]

Since \( a_\theta = 0 \) we have, (compare (1) and (4))

\[
F = \frac{\lambda^2 m}{Br^2} u_T, \quad \text{the Inverse Square Law.}
\]

We have shown that the Kepler's First and Second Laws imply the Inverse Square Law. The above calculation is not what Newton did. Something closer to what he did is outlined in Exercise 1 where it is shown that the Second and Third Laws imply the Inverse Square Law. In any case, this section shows that experimental evidence well used can lead to nice and powerful theoretical results.

---

**Exercise 1.** Assume, as Newton did, that the moon is in a circular orbit and that Kepler's Second and Third Laws hold. Show that \( \theta' \) is a constant and therefore it is uniform circular motion. Introduce the linear speed \( v = ds/dt \) along the circle. Show that \( a_T = -(v^2/r) \). This is the usual formula for centripetal acceleration. Combine with the Third Law to get the Inverse Square Law.

**Exercise 2.** Justify the steps in the following without doing any integrals.

\[
\int_{-a}^{a} 2\sqrt{b^2 - u^2} \, du = \int_{-b}^{b} 2\sqrt{1 - u^2} \, du = ab
\]

Why does this show that the area of an ellipse is \( \pi ab \)? We need this formula for the area of an ellipse in deriving the Third Law. Hint: Evaluate the last integral by interpretation rather than calculation.
6. KEPLER'S LAWS AS CONSEQUENCES OF NEWTON'S LAWS

If one decides that $F = mA$ and the Inverse Square Law are correct, do Kepler's Laws follow? This is an important question. Kepler's Laws were empirical results based on fanciful hope and data which had unavoidable inaccuracies. Suppose one can conduct other experiments which verify the Inverse Square Law. If Kepler's Laws are a consequence of the Inverse Square Law, then Kepler's Laws will no longer be empirical results from a single set of datum. A physical theory is strengthened by logical implications between various empirical results. In this section we will show that Kepler's Laws can be deduced from the Inverse Square Law.

First we give an argument that for any central force field, the motions are planar. Suppose that the planet is at a point $P$ and that its motion has a tangent vector $T$ at that point. Draw the plane through the sun, through $P$, and containing the tangent vector $T$. If $Z$ is the coordinate normal to this plane, then the force and hence the acceleration in the $Z$ direction is zero. That is $Z(t_0) = 0$, $Z'(t_0) = 0$ and $Z''(t) = 0$ for all $t$. This implies $Z(t) \equiv 0$; the motion is planar.

Secondly, if the force is a central force field we show Kepler's First Law holds. In the planar polar coordinates, $a_\theta = 0$. Thus we may start at equation (9) and retrace our steps to (8) and (7) giving the equal area results.

Exercise 3. Do it.

To get the Second and Third Laws of Kepler we must specialize the central force field to the Inverse Square Law. We have
(12) \[ r'' - r(\theta')^2 = -\frac{GM}{r^2} \]

and of course (8). The solution of these differential equations is difficult, but is made easier by looking at the goal. We want \( r \) to be a conic as in (10). Notice that \( 1/r(\theta) \) is very simple. This suggests introducing the function \( w(\theta) = 1/r(\theta) \) and deriving a differential equation that it satisfies. We have

\[
\frac{dw}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta} = -\frac{1}{r^2} \frac{r'}{\theta'} = -\frac{r'}{\lambda} \quad \text{using (8).}
\]

Then

\[
\frac{d^2w}{d\theta^2} = -\frac{d}{dt} \left( \lambda \right) \frac{dt}{d\theta} = -\frac{r''}{\lambda \theta'} = -\frac{1}{\lambda \theta'} \left( (r(\theta'))^2 \cdot \frac{GM}{r^2} \right)
\]

\[
= -\frac{r^{(b)}}{\lambda} + \frac{GM}{\lambda r^2 \theta'} = -w + \frac{GM}{\lambda^2} \quad \text{using (12).}
\]

Then from (8)

(13) \[ \frac{d^2w}{d\theta^2} + w = \frac{GM}{\lambda^2} \]

It turns out that every solution of (13) is given in the form

\[ w(\theta) = \frac{GM}{\lambda^2} + \beta \cos(\theta + \alpha) \]

for some constants \( \beta \) and \( \alpha > 0 \) (See Appendix 2).

Thus

(14) \[ r(\theta) = \frac{B}{1 + e \cos(\theta + \alpha)} \]

where

\[ B = \frac{\lambda^2}{GM} \quad \text{and} \quad e = \frac{\beta \lambda^2}{GM} \]

Hence any such motion describes a conic section.

Notice that parabolas and hyperbolas are possible. In fact the constants \( \beta \), \( \alpha \) depend on some initial condition. If they are selected so that \( e \geq 1 \), then the mass leaves
the solar system. The only bounded orbits are therefore ellipses (and circles as a special case). This is Kepler's Second Law.

To derive Kepler's Third Law we need to compute the period $T$ and semi-major axis $a$ of an ellipse. Let $b$ be the semi-minor axis and $c = \sqrt{a^2 - b^2}$ be the focal length.

The formula for $T$ is easy. Since the area of an ellipse is $\pi ab$, (see Exercise 2) we have by (7) and (8)

$$\pi ab = \int_0^T \frac{1}{2} r^2 \theta' dt = \frac{1}{2} T,$$

or

$$T^2 = \frac{4\pi^2}{\lambda^2} a^2 b^2 \ .$$

(15)

![Figure 4. An ellipse with the focus at the origin. This is the picture that Kepler saw in his mind's eye. It is one we can describe neatly in polar coordinates.](image)

The major axis of an ellipse with ends $P$ and $P'$ has length $2a$. The length $PF$ is the minimum distance from $F$ to the ellipse, $P'F$ is maximum, while the focal distance $c$ from the center $C$ is $c = 1/2|PP'| = |PF| - |PF| = a - |PF| .

Looking at (14) we see that $r$ is smallest when $\cos(\theta + \alpha) = 1$ and largest when $\cos(\theta + \alpha) = -1$. Thus

$$a = \frac{1}{2} \left( \frac{B}{1 + e} + \frac{B}{1 - e} \right) = \frac{B}{1 - e^2} \ .$$

(16)
Moreover

\[ b^2 = a^2 - c^2 = a^2 - \left( a - \frac{B}{1 + e} \right)^2 \]

\[ = 2a \frac{B}{1 + e} - \frac{B^2}{(1 + e)^2} \]

Using (16)

\[ \frac{B^2}{(1 + e)^2} = \frac{B}{1 + e} \ a(1 - e) . \]

Thus

\[ b^2 = \frac{Ba}{1 + e} \left[ 2 - (1 - e) \right] = Ba . \]

Putting this into (15) gives

(17) \[ T^2 = \frac{4\pi^2}{\lambda^2} \frac{Ba}{\lambda^2} = \frac{4\pi^2}{\lambda^2} \frac{\lambda^2}{GM} a^3 = \frac{4\pi^2}{GM} a^3 \]

But \(4\pi^2/\lambda^2\) is a constant that only depends on the units and the mass of the sun, a Solar System Constant! This is Kepler's Third Law.

7. COMMENTS

We have discussed the relationship between Kepler's Laws and the Inverse Square Law in the context of two masses. We also assumed that the larger mass was fixed. In fact, it will wobble slightly. What is fixed is the common center of mass. When more than two masses are involved, exact description of the motions can be ascertained only in very special cases. This is called the n-body problem. It is the focus of a great deal of mathematical research.
8. FURTHER EXERCISES

Exercise 4. Suppose you observe that for the earth $a = 1.495 \times 10^8$ km. and $T = 365.25$ days. If $G = 6.670 \times 10^{-11}$ in $m^3/kg\cdot sec^2$ find the mass of the Sun.

Exercise 5. If the eccentricity of the earth's orbit is given to be 0.0167322, and $a$ as in Exercise 4 find the exact equation of the earth's orbit.

Exercise 6. You know that the usual expression for gravity at sea level is $g = -9.807$ m/sec$^2$. Use the Inverse Square Law and the fact that we may replace the earth by a point mass at its center to get an exact expression for the Inverse Square Law with the earth being the large mass. Use 6371 km. as the radius of the earth. Find the mass of the earth. Hint: Compare the two formulae at the surface of the earth.

Exercise 7. If a satellite going around the earth remains in a circular orbit, then centripetal acceleration must balance the acceleration due to gravity. Using this equation, compute the linear speed $v$ the satellite must have if its altitude is 100 miles above sea level. (Ignore air resistance.) 1 mile = 1609.35 meters.

Exercise 8. The Intelsat series geosynchronous satellite remains above a fixed point on the earth's equator. If it is in a circular orbit you can deduce its altitude. Please do.

Exercise 9. OSO 4, launched October 18, 1967 was in an orbit with radius between $5.375$ and $5.697 \times 10^5$m. above sea level. What is the period of its motion?

Exercise 10. Take the general formula $ma = f(r)$ with $\theta'$ removed by (8). Multiply by $r'$ and integrate once. Then replace $\lambda$ by (8) and derive the formula

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 + G(r) = \text{constant}$$

where $dG/dr = -f$. (You may recall the formula for $ds$ in polar coor-
9. HINTS TO THE SOLUTIONS OF THE EXERCISES

Exercise 1. From (8) $s^\prime = \lambda/r^2$ is a constant. For a circle $s = r\theta$ so $v = ds/dt = r\theta^\prime = \lambda/r$. Since $r'' = 0$ $a_r = -r(\theta^\prime)^2 = -v^2/r$. From $2\pi r = rT$ and Kepler's Third Law $a_r = C/r^2$.

Exercise 2. The last integral is the area of a unit circle.

Exercise 4. Solve (17) for $M$.

Exercise 5. Use (16) to compute $B$.

Exercise 6. Write $f(r) = mC/r^2$, $f(6371 \times 10^8) = -mg$. Compare with (2) to find $M$.

Exercise 7. The equation $C/r^3 = v^2/r$ can be solved for $v$ in terms of known quantities.

Exercise 8. Compute $v = \omega r$, $\omega$ the rate of spin of the earth and apply the equation in Exercise 6 to find $r$, or use (17).

Exercise 9. Apply the derivation of (16) to find $a$, and then use (17).

Exercise 10. $ds^2 = dr^2 + r^2 d\theta^2$

Exercise 11. Spin a weight at the end of a string. Measure the tug of the string. Keeping $r$ fixed, double the speed, remeasure the tug. Keeping $v$ fixed, double $r$. 

Exercise 12. Describe a laboratory experiment that would show that centripetal acceleration is $K(v^2/r)$. Newton did this with a thought experiment. He reasoned that the moon must "fall" in its circular orbit in one second the same distance as if it were dropped from a stationary position.
10. MODEL EXAM

1. Suppose we are considering a planet's motion around the Sun.
   a) At what point on the orbit is the vector from the Sun to the planet turning the most rapidly? (First Law)
   b) Recalling that speed $v = ds/dt$, find a formula for the speed of the planet which involves the polar coordinates $r$ and other constants, but not $\theta$. (Second and First Laws).
   c) On the basis of b), when is the speed the maximum?

2. Using the Third Law (or otherwise) find the relationship between the period of a satellite in a circular orbit and its distance from the center of the earth.

3. Describe why Kepler's Laws were more acceptable after Newton's work.

4. Given that the acceleration due to gravity at sea level on the earth is $g = -32 \text{ ft/sec}^2$. Find the constant in the Inverse Square Law in the units feet and seconds. Use the radius of the earth as 4000 miles and 5280 feet in 1 mile.

5. Suppose that the gravitational force were $F = k/r^2$. Would it still be true that in a satellite's motion, the radius vector sweeps area out in a constant rate?
Conics in Polar Coordinates. If one of the foci is at the origin then the equation of any conic section in polar coordinates is of the form

\[ r = \frac{b}{1 + e \cos(\theta + \alpha)} \]

The number \( e \) is called the eccentricity and the various conic sections are given by the table:

- \( e = 0 \): circle
- \( 0 < e < 1 \): ellipse
- \( e = 1 \): parabola
- \( e > 1 \): hyperbola

The table is easy to remember. If \( e < 1 \) then the denominator is never zero and \( r \) is bounded so we have an ellipse. If \( e = 1 \) then the denominator is zero precisely once as \( \theta \) traverses a complete rotation, the point \((r, \theta)\) "jumping across the open end of the parabola at infinity." If \( e > 1 \), then \( r \) is infinite twice "jumping from one branch of the hyperbola to the other" each time.

APPENDIX 2

Solutions of Certain Differential Equations. We want to show that the differential equation

\[ * \quad v'' + v = D \]

with \( v(0) = a_0 \), \( v'(0) = a_1 \) has a unique solution.

Suppose \( v_1 \) and \( v_2 \) are two solutions and let \( h \) (\( h \) for a function pulled out of a hat) be defined by

\[ h(\theta) = (v_1 - v_2)^2 + (v_1' - v_2')^2 \]

Then

\[ \frac{1}{2} h'(\theta) = (v_1 - v_2)(v_1 - v_2)' + (v_1' - v_2')(v_1'' - v_2'') \]

\[ = (v_1' - v_2')(v_1 - v_2) + (v_1'' - v_2'') \]
\[ = (v'_1 - v'_2)[D - D] = 0 \quad \text{(Using } * \text{ twice)} \]

so \( h \) is a constant. But \( h(0) = 0 \) means \((v'_1 - v'_2)^2 + (v'_1 - v'_2)^2 \equiv 0\). Thus \( v_1 - v_2 \equiv 0 \). That is \( v_1 = v_2 \).

Now one solution of the problem is (just plug it in)

\[ v(\theta) = D - \frac{a_1}{\sin \alpha} \cos(\theta + \alpha) \]

with

\[ \cot \alpha = \frac{D - a_0}{a_1} \]

if \( a_1 \neq 0 \). If \( a_1 = 0 \) take \( v(\theta) = D - (D - a_0) \cos \theta \).

Since every solution of the differential equation satisfies the extra conditions at \( 0 \) for some \( a_0 \) and \( a_1 \), it is of the form \( v = D + \beta \cos(\theta + \alpha) \). To make \( \beta > 0 \) replace \( \alpha \) by \( \alpha + \pi \) since \( \cos(\theta + \alpha + \pi) = -\cos(\theta + \alpha) \) changes the sign of \( \beta \).