Applications of Calculus in Geometrical Probability

Richard M. Dahlke
Robert Fakler
Title: Applications of Calculus in Geometrical Probability

Authors: Richard M. Dahlke
Robert Fakler
University of Michigan-Dearborn
Department of Mathematics
4901 Evergreen Road
Dearborn, MI 48128

Mathematics Field: Calculus

Applications Field: Probability, problem solving, computer simulation

Target Audience: Students with knowledge of single or double integration.

Abstract: The geometrical probability model is defined, and examples of probability problems are given in which considerable thought is required in translating the problems into mathematical models. Once models are constructed, probabilities are derived by using various methods for calculating areas or volumes of the sample space and success region, using single or double integration. A computer simulation of a probability problem is motivated, defined, and used to solve probability problems already solved using analytical techniques. Exercises are given after each of the three major sections of the unit.

Prerequisites: Most of the unit only requires a knowledge of single integration. Double integration is a prerequisite for several examples and exercises near the end of the unit.


© Copyright 1989 by COMAP, Inc. All rights reserved.
Applications of Calculus in Geometrical Probability

Richard M. Dahlke
Robert Fakler
Department of Mathematics
University of Michigan-Dearborn
Dearborn, MI 48128

Table of Contents

1. INTRODUCTION ............................................. 1
   1.1 What is Geometrical Probability? ................. 1
   1.2 Why Study It? ....................................... 1
2. GEOMETRICAL PROBABILITY MODEL .................. 2
   2.1 Geometrical Probability Model ................. 2
   2.2 Determining Probability Bounds ............. 3
3. APPLICATIONS ........................................... 4
   3.1 Functions of One Variable (Single Integration) ............ 4
   3.2 Exercises .......................................... 18
   3.3 Functions of Two Variables (Double Integration) ............. 19
   3.4 Exercises .......................................... 23
4. SIMULATION ............................................. 25
   4.1 Real-World Simulation ........................... 25
   4.2 Computer Simulation ............................... 25
   4.3 Exercises .......................................... 28
5. SAMPLE EXAM ........................................... 29
6. ANSWERS TO EXERCISES .................................. 29
7. ANSWERS TO SAMPLE EXAM ........................... 30
   ABOUT THE AUTHORS .................................... 30
MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop, through a community of users and
developers, a system of instructional modules in undergraduate mathematics
and its applications to be used to supplement existing courses and from
which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathemati-
cians, scientists, and educators. UMAP was funded by a grant from the
National Science Foundation and is now supported by the Consortium for
Mathematics and Its Applications (COMAP), Inc., a non-profit corporation
engaged in research and development in mathematics education.

COMAP STAFF

Paul J. Campbell         Editor
Solomon A. Garfunkel     Executive Director, COMAP
Laurie W. Aragon         Business Development Manager
Philip A. McGaw          Production Manager
Evelyn Moore             Copy Editor
Roland Cheyney           Project Manager
Dale Jolliffe            Production Assistant
Annemarie S. Morgan      Administrative Assistant
John Gately              Distribution
1. Introduction

1.1 What is Geometrical Probability?

Geometrical probability deals with probability on infinite sample spaces for which each outcome of the experiment in question is equally likely to occur. It allows us to calculate the probability of an event occurring at random in a single trial of an experiment by identifying the sample space with a geometric region \( R \) and the event with a subregion \( r \) of \( R \). We can then use geometry to find the desired probability.

1.2 Why Study It?

Geometrical probability is replete with applications of high-school and lower-division college mathematics. In this module we focus on problems in geometrical probability whose solutions require the use of calculus—in particular, integration. The problems provide an interesting and challenging alternative to the usual problems presented as applications of integration. Considerable thought is required in translating the probabilistic situation into a mathematical model. Making such translations lies at the very foundation of problem solving. Once the mathematical model is constructed, the answer is derived by using an interesting assortment of precalculus content and methods for calculating areas or volumes using integration.

Other important benefits can be derived by studying geometrical probability. For example, this type of probability deals with a continuous random variable, as opposed to the probability of a discrete random variable, which you may have studied in high school. Upper-division college probability and statistics courses dealing with continuous random variables are variations of geometrical probability. Unfortunately, these courses are very abstract, even at the beginning; hence, most students do not gain an intuitive view of the subject. Geometrical probability goes a long way toward providing this necessary intuition. A particularly nice feature of the subject is that its definition is very intuitive and can be presented in a short time, thus allowing you to begin working on significant problems almost immediately.

We also give some interesting variations of some of the problems by changing one or more of the problem conditions or the question that is asked. (Just changing one or more numbers in the problem can lead to some interesting problem solving.) In some instances, a
variation may be a generalization of a problem. Examples of problem variations are given, so that you are prepared for offering problem variations in the exercises. You probably have never been provided with the opportunity to do this; when you do it, you are operating very much like a mathematician. This flexibility adds a whole new dimension to your mathematics education. Your experiences in calculus to date should have made you aware of how computationally-oriented calculus courses are, frequently at the expense of higher-level learning objectives. Posing problem variations and then solving them is just another way in which this Module can foster your higher-level thinking skills.

Near the end of the module we provide you with alternatives to determining geometrical probabilities by mathematical methods, namely, real-world simulation, and computer simulation. The first method eliminates the need for constructing a mathematical model of the problem, and the second eliminates the need for solving a mathematical model of the problem, e.g., you can bypass evaluating (possibly difficult integrals, needed for an analytical solution).

2. Geometrical Probability Model

2.1 Geometrical Probability Model

What is geometrical probability? Simply put, for any real-world experiment with equally likely outcomes, we identify the outcomes of the experiment with points in a geometric region (one-, two-, or three-dimensional), called the sample space of the experiment and denoted $R$. Then the real-world experiment corresponds to the mathematical experiment of randomly choosing a point in the geometric region $R$. If some event $E$ in the sample space of our real-world experiment is specified, it corresponds to a subset $r$ (called event $r$) of the geometric region $R$ that represents the sample space of this experiment. With a given event $r$ in mind, we say an outcome of the experiment is a success if it belongs to the event $r$ and a failure if it does not. These notions are summarized in Figure 1.

This mathematical experiment of choosing a random point in a geometric region is the mathematical model of the real-world experiment. We will use this model to determine our probabilities.

If we want the probability that event $E$ in the real-world experiment will occur, we are asking for the probability $p(r)$ that a randomly-chosen point in $R$ will belong to $r$. It seems reasonable
that, in the one-, two- and three-dimensional cases, we should have
\[ p(r) = \frac{\text{area of } r}{\text{area of } R}, \frac{\text{length of } r}{\text{length of } R}, \text{ or } \frac{\text{volume of } r}{\text{volume of } R}, \] respectively.

(For example, if the area of \( r \) is one-half the area of \( R \), we would expect, in the long run, a randomly-chosen point in \( R \) to be in \( r \) one-half of the time.)

To summarize, we say that the probability of a randomly chosen point in \( R \) belonging to \( r \) is
\[ p(r) = \frac{\text{measure of } r}{\text{measure of } R}. \]

### 2.2 Determining Probability Bounds

Since event \( r \) is a subregion of sample space \( R \), we have
\[ 0 \leq \text{measure of } r \leq \text{measure of } R. \]

Thus, dividing the members of these inequalities by the measure of \( R \) (which is assumed to be positive), gives
\[ \frac{0}{\text{measure of } R} \leq \frac{\text{measure of } r}{\text{measure of } R} \leq \frac{\text{measure of } R}{\text{measure of } R}. \]

That is, \( 0 \leq p(r) \leq 1 \). This inequality shows that geometrical proba-
Probabilities are bounded below by 0 and above by 1. Notice that for 
\( p(r) = 0 \), the measure of \( r \) must be 0 (a point has length 0, a curve 
has area zero, and a planar region has volume 0); and for \( p(r) = 1 \),
the measure of \( r \) must equal the measure of \( R \). Also, the area 
(volume) of a region minus its boundary is equal to the area 
(volume) of that region with the boundary included. For example, if 
\( r \) is the set of points interior to a circle, and \( R \) consists of these points 
as well as the points on the circle, then the area of \( r \) equals the area 
of \( R \). In the problems in this Module, probabilities will not be 
changed if, in the inequalities that appear in the mathematical 
models, the symbol \( < \) or \( > \) is replaced by \( \leq \) or \( \geq \), respectively, 
and vice versa.

We now present some applications that require the use of 
integration to determine the areas of success regions. The sample 
spaces are all two-dimensional.

3. Applications

3.1 Functions of One Variable
(Single Integration)

A football fan can readily relate to our first example, since it 
deals with the probability of blocking a field-goal attempt, assuming 
certain idealized conditions. The solution shown here requires the use 
of the law of sines, several trigonometric identities, graphing a 
trigonometric inequality containing the sum of a sine and cosine 
function, finding a critical point, and integrating the sum of a sine 
and cosine function.

Example 1 (Place-Kicker Problem). A defensive tackle is trying to 
block a field-goal kick. (See Figure 2a.) Suppose that the ball is 
kicked at an angle of \( \pi/6 \) and \( h \) is the distance from the tackle’s 
feet to the tip of his outstretched fingers. If, when the ball is 
kicked, the tackle’s feet are at a random distance between 0 and 
\( 2h \) from the kicker and the tackle is falling forward so that his 
body makes a random angle \( \alpha \) between 0 and \( \pi/2 \) with the 
ground, what is the probability that the kick will be blocked?
(Assume that the tackle’s body lies in the same plane as the path 
of the kick.)

Solution. Our situation is modeled in Figure 2b, in which the tackle’s 
feet are at \( A \), the ball is at \( B \), side \( AC \) represents the body of the 
tackle, angle \( \theta \) \( (= \pi/6) \) is the angle at which the ball is kicked,
and $\beta$ is the angle with a horizontal line that the line makes from where the ball is placed to the end of the tackle's outstretched arms.

We have a success (i.e., the ball is blocked) if $\beta > \pi/6$. It turns out that the solution to the problem doesn't depend on $h$. Therefore, to simplify matters, we choose the convenient value of 1 for $h$. Thus, we will work with Figure 3.

Let $x =$ length of side $AB$. The possible positions of the defensive tackle can be represented by the ordered pair $(\alpha, x)$, where $0 < x < 2$ and $0 < \alpha < \pi/2$. We now find a relationship between $x$ and $\beta$ and then use the knowledge that we have a success if
\( \beta > \pi/6 \). By the law of sines,

\[
\frac{\sin \beta}{1} = \frac{\sin(\pi - \alpha - \beta)}{x}
\]

Therefore,

\[
x \sin \beta = \sin(\pi - \alpha - \beta)
\]

\[
= \sin \pi \cos(-\alpha - \beta) + \cos \pi \sin(-\alpha - \beta)
\]

\[
= \sin(\alpha + \beta)
\]

\[
= \sin \alpha \cos \beta + \cos \alpha \sin \beta.
\]

This gives

\[
x = \sin \alpha \cot \beta + \cos \alpha
\]

and

\[
\cot \beta = \frac{x - \cos \alpha}{\sin \alpha}.
\]

Since \( \beta > \pi/6 \) if and only if \( \cot \beta < \sqrt{3} \), we have a success if

\[
\frac{x - \cos \alpha}{\sin \alpha} < \sqrt{3}
\]

or

\[
x < \sqrt{3} \sin \alpha + \cos \alpha.
\]
The region representing the sample space of the experiment is the graph of $S = \{(\alpha, x) | 0 < \alpha < \pi/2 \text{ and } 0 < x < 2\}$, and the shaded success region is the graph of $\{(\alpha, x) \in S | x < \sqrt{3} \sin \alpha + \cos \alpha\}$. These regions are shown in Figure 4. (The maximum value of $x$ in the success region was found to be 2 by finding the critical point $\pi/3$.) The area of the sample space is $\pi$ and the area of the success region is

$$\int_{0}^{\pi/2} (\sqrt{3} \sin \alpha + \cos \alpha) \, d\alpha = 1 + \sqrt{3}.$$ 

Hence,

$$p = \frac{\text{area of success region}}{\text{area of sample space}} = \frac{1 + \sqrt{3}}{\pi} = 0.87.$$ 

The next example is interesting from many standpoints, not the least of which are its many variations, some of which will be listed after the solution given here, and one which will be stated and solved as Example 6.

**Example 2 (Die Problem).** A die is randomly thrown onto a board ruled with parallel lines, with the distance between adjacent lines
being greater than the length of a diagonal of a face of the die. Find the probability that the die will cross a line.

Solution. Let $a$ be the distance between adjacent lines and $s$ the length of an edge of the die. Hence $a > s\sqrt{2}$. Now, each outcome of the experiment can be represented by an ordered pair $(\theta, x)$, as shown in Figure 5. In the figure, $x$ is the distance between the center of the die face and the nearest line to the right of it; and $\theta$ is the angle whose vertex is at the center of the square die face, with one side perpendicular to the parallel rulings, and the other side containing the vertex of the die face that is furthest to the right.

The rectangular region $\{(\theta, x) | -\pi/4 \leq \theta \leq \pi/4, 0 \leq x \leq a\}$ represents the sample space of the experiment. We have a success if either

$$x < \frac{s\sqrt{2}}{2} \cos \theta \quad (0 \leq x \leq a/2)$$

or else

$$a - x < \frac{s\sqrt{2}}{2} \cos \theta \quad (a/2 \leq x \leq a).$$
The sample space of the experiment and the shaded subregion representing successes are shown in Figure 6.

Since cosine is an even function, the area of the success event is

\[ 2 \int_{\pi/4}^{\pi/2} \left( \frac{\sqrt{2}}{2} \right) \cos \theta \, d\theta = \sqrt{2} \sin \theta \bigg|_{\pi/4}^{\pi/2} = 2s. \]

The area of the sample space is \( \pi a/2 \). Hence

\[ p = \frac{2s}{\pi a/2} = \frac{4s}{\pi a}. \]

Notice from the formula that, with \( s \) fixed, \( p \to 0 \) as \( a \to \infty \) (why is this quite plausible?). Now, for \( a = 4 \) and \( s = 1 \), \( p = 1/\pi \), and the reciprocal of \( p \) is \( \pi \). Hence an empirical probability for this problem can be used to approximate \( \pi \) (assuming that the mathematical model we constructed is a good model of this probability experiment.) To do this, toss a die, say a few hundred times, onto a sheet ruled into lines that are a distance apart of 4 times the length of the die's edge. Then form the ratio of the number of times the die crossed a line, divide by the total number of tosses, and then take the reciprocal of this. (You are encouraged to perform this experiment and compare the probability found with \( \pi \).)

Two variations to the Die Problem are now stated. (Note: Example 6 is also a variation.) You are encouraged to solve them and to come up with other variations and their solutions.
Variation 2A. Suppose for Example 2 that the condition \( a > \sqrt{2} s \) is changed to \( s < a < \sqrt{2} s \). What changes would have to be made in the solution above to solve this problem? Solve it. Show that, for \( s \) fixed, as \( a \to (\sqrt{2} s)^+ \), \( p \to 2\sqrt{2} / \pi \). Use the solution to Example 2 to show that this is also the case as \( a \to (\sqrt{2} s)^- \).

Variation 2B. Solve Example 2 in the case that the "die" is a tetrahedron with all faces being congruent equilateral triangles of side \( s \), and that the distance between the parallel rulings is greater than a side of the triangle.

Variation 2C. Solve Example 2 in the case that a thin rectangle is tossed with side \( m \) larger than side \( n \), and the width between the rulings is greater than a diagonal of the rectangle.

Variation 2D. See Example 6.

The theory of local extrema plays a key role in obtaining the success region for the next example. The real challenge is determining this region. Once this is done, determining the necessary area is an easier task.

Example 3 (Roots of Cubic Equation). Find the probability that all the roots of the equation \( x^3 + 3ax + 2b = 0 \) are real, where \( a \) and \( b \) are randomly chosen from \([-3,3]\) and \([-8,8]\), respectively.

Solution. For the function to have three real roots, it must have a nonnegative local maximum and a nonpositive local minimum. (Do you see why?) Now,

\[
    f'(x) = 3x^2 + 3a = 0
\]

for \( x = \pm \sqrt{-a} \). Thus, for a success (i.e., three real roots), we must have critical points; and this is possible only if \( -a > 0 \). We apply the second derivative test for local extrema to the critical points \( \pm \sqrt{-a} \):

\[
    f''(x) = 6x,
\]

so

\[
    f''(\sqrt{-a}) = 6\sqrt{-a} > 0
\]

and

\[
    f''(-\sqrt{-a}) = -6\sqrt{-a} < 0.
\]

Hence, for \( -a > 0 \), \( x = \sqrt{-a} \) gives a local minimum, and \( x = -\sqrt{-a} \) gives a local maximum. Therefore, we have a success
if \( a < 0 \), \( f(\sqrt{-a}) \leq 0 \), and \( f(-\sqrt{-a}) \geq 0 \). (Our subgoal here is to find relationships between \( a \) and \( b \) such that all \((a, b)\) satisfying these relationships belong to the success region.) Now \( f(\sqrt{-a}) \leq 0 \) implies
\[
(\sqrt{-a})^3 + 3a\sqrt{-a} + 2b \leq 0
\]
\[
(-a)^{3/2} \leq 3(-a)^{3/2} + 2b \leq 0
\]
\[
-2(-a)^{3/2} + 2b \leq 0
\]
\[
b \leq (-a)^{3/2},
\]
and \( f(-\sqrt{-a}) \geq 0 \) implies
\[
(-\sqrt{-a})^3 + 3a(-\sqrt{-a}) + 2b \geq 0
\]
\[
-(-a)^{3/2} + 3(-a)^{3/2} + 2b \geq 0
\]
\[
2(-a)^{3/2} + 2b \geq 0
\]
\[
b \geq -(-a)^{3/2}.
\]
The sample space is the rectangular region
\[
S = \{(a, b) | a \in [-3, 3] \text{ and } b \in [-8, 8]\},
\]
and the success region is \( \{(a, b) \in S | a < 0, \ b \leq (-a)^{3/2} \text{ and } b \geq -(-a)^{3/2}\} \). These regions are shown in Figure 7.

![Figure 7](image-url)
The area of the sample space is \(4(3)(8) = 96\), and the area of the success region is

\[
2 \int_{-3}^{0} (-a)^{3/2} \, da = -\left(4(-a)^{5/2}\right) \bigg|_{-3}^{0} = 4(3)^{5/2}/5 = 12.47.
\]

Hence

\[
p = 12.47/96 = 0.13.
\]

**Variation 3A.** Find the probability asked for in Example 3 if \(a \in [-A, A], \ b \in [-B, B]\), and \(B^{2/3} < A\).

**Variation 3B.** If \(a\) and \(b\) are chosen randomly in the intervals \([-1, 4]\) and \([-3, 6]\), respectively, find the probability that \(ax^2 + 2x + b = 0\) has no real root.

The success region for the following example is the union of four regions, each of which is bounded on one side by a hyperbola. The region is found by constructing inequalities and their graphs. The integration to find the area of the success region yields the natural logarithm function.

**Example 4 (Dividing a Field).** Mr. I. M. Rich has four sons who will inherit a valuable parcel of real estate when he dies. The parcel is a field 1 mile by 1 mile square. Mr. Rich’s will specifies that, to divide this property among his sons, a location in the field will be randomly chosen and the field will be divided into 4 rectangular parts by lines through the location parallel to the sides. What is the probability that one of the 4 sons will receive less than one-eighth of the field?

**Solution.** Let the randomly-chosen point be represented by \((x, y)\).

Then the sample space of the experiment is the square region \(\{(x, y)|0 \leq x \leq 1\ \text{and}\ 0 \leq y \leq 1\}\). To determine which outcomes will be successes, we first divide the square region into four square subregions with lines through the center of the square region parallel to its sides. If we choose a point \((x, y)\) in the square region and divide it into rectangles by lines through \((x, y)\) parallel to its sides, we can then tell which of the four rectangles will have the smallest area if we know which of the four square subregions \((x, y)\) lies in. The four possible cases are shown in Figure 8.

The conditions that must be met for a success are as follows:

**Case 1.** \(1/2 \leq x \leq 1, \ 1/2 \leq y \leq 1\). The area of the shaded rectangle is \((1-x)(1-y)\). Therefore, we must have \((1-x)(1-y) < 1/8\).
Figure 8. Four cases for dividing the field for Example 4.

Case 2. $0 \leq x \leq 1/2$, $1/2 \leq y \leq 1$. The area of the shaded rectangle is $x(1-y)$. Therefore, we must have $x(1-y) < 1/8$.

Case 3. $0 \leq x \leq 1/2$, $0 \leq y \leq 1/2$. The area of the shaded rectangle is $xy$. Therefore, we must have $xy < 1/8$.

Case 4. $1/2 \leq x \leq 1$, $0 \leq y \leq 1/2$. The area of the shaded rectangle is $(1-x)y$. Therefore, we must have $(1-x)y < 1/8$.

The square region representing the sample space of the experiment and the shaded subregion representing successes are shown in Figure 9.

The area of the success region is

$$4 \left[ (1/4)(1/2) + \int_{1/4}^{1/2} \frac{1}{8x} \, dx \right] = (1/2) \left[ 1 + \int_{1/4}^{1/2} \frac{1}{x} \, dx \right] = (1/2)(1 + \ln 2) = 0.847.$$ 

In the next problem, the law of cosines is used to formulate the inequalities that describe the success region. This interestingly-shaped
region turns out to be the intersection of the solution sets of three quadratic inequalities in two variables. Determining its area is fairly challenging.

**Example 5 (Acute-Angled Triangle).** Linda is a mathematician who enjoys making bets with her friends on matters involving chance happenings. Unfortunately for her nonmathematician friends, Linda has an advantage because of her ability to solve geometrical probability problems. Suppose she came to you with a rod (the length doesn't matter) in her hand and was willing to give you $10,000 if, after the rod was sawed (theoretically) into three pieces by randomly choosing two points at which to make the cuts, an acute-angled triangle could be formed with the pieces. If it were not acute-angled, then you would have to pay Linda $2500. Is this a smart bet on your part?

**Solution.** To answer the question, we first need to calculate the probability of obtaining an acute-angled triangle. Let the rod be of length $a > 0$ and represent it by the interval $[0, a]$, and let $x$ and $y$ be the coordinates of the two “saw” points. We will assume that $x < y$; hence, $0 < x < y < a$. These relationships are illustrated in Figure 10.
We can form a triangle if and only if the following inequalities hold simultaneously:

\[ x < (y - x) + (a - y), \]
\[ y - x < x + (a - y), \]
\[ a - y < x + (y - x). \]

Therefore, \( x < (1/2)a, \ y - x < (1/2)a, \) and \( y > (1/2)a. \) The triangle formed by these segments is shown in Figure 11.

From the law of cosines we get

\[
\cos A = \frac{x^2 + (a - y)^2 - (y - x)^2}{2x(a - y)},
\]
\[
\cos B = \frac{x^2 + (y - x)^2 - (a - y)^2}{2x(y - x)},
\]
\[
\cos C = \frac{(a - y)^2 + (y - x)^2 - x^2}{2(a - y)(y - x)}. \]

Since \( A, B, \) and \( C \) are all to be acute angles, we must have \( \cos A > 0, \cos B > 0, \) and \( \cos C > 0. \) Therefore, the numerators in the fractions on the right-hand side of each of the above equations must be positive; and so

\[ x^2 + (a - y)^2 - (y - x)^2 > 0, \]
\[ x^2 + (y - x)^2 - (a - y)^2 > 0, \]
\[ (a - y)^2 + (y - x)^2 - x^2 > 0. \]
Figure 12. Sample space and success region for Example 5.

Figure 13. Drawing to aid in calculating the area of the success region for Example 5.
Simplifying, we get

\[ y < \frac{a^2}{2(a - x)}, \quad y > \frac{2x^2 - a^2}{2(x - a)}, \quad \text{and} \quad x < \frac{a^2}{2y} + y - a, \]

respectively. The triangular region representing the sample space of the experiment and the shaded subregion representing successes are shown in Figure 12. The area of the region representing the sample space is \(a^2/2\). To calculate the area of the success region, consider Figure 13. The area of the success region is

\[
\text{area of } \triangle QNM - \text{area of region } R_1 - \text{area of region } R_2 - \text{area of region } R_3.
\]

Now, the area of \(\triangle QNM\) is \((1/2)(a/2)^2 = a^2/8\), and

area of \(R_1 = \int_0^{a/2} \left( \frac{(2x^2 - a^2)}{2(x - a)} - a/2 \right) \, dx \]

\[= \left[ \frac{x^2}{2} + ax/2 + (1/2)a^2 \ln|x - a| \right]_0^{a/2} \]

\[= 3a^2/8 - (1/2)a^2 \ln 2; \]

area of \(R_2 = \int_0^{a/2} \left( a/2 - \frac{a^2/2}{y} + y + a \right) \, dy \]

\[= \left[ 3ay/2 - (1/2)a^2 \ln y - y^2/2 \right]_0^{a/2} \]

\[= 3a^2/8 - (1/2)a^2 \ln 2; \]

area of \(R_3 = \int_0^{a/2} \left( x + a/2 - \frac{a^2/2(a - x)}{2(a - x)} \right) \, dx \]

\[= \left[ \frac{x^2}{2} + ax/2 + (1/2)a^2 \ln|a - x| \right]_0^{a/2} \]

\[= 3a^2/8 - (1/2)a^2 \ln 2. \]

Therefore, the area of the success region is

\[a^2/8 - \text{area of } R_1 - \text{area of } R_2 - \text{area of } R_3 \]

\[= (1/2)a^2(3 \ln 2 - 2),\]

and

\[p = \frac{(1/2)a^2(3 \ln 2 - 2)}{a^2/2} = 3 \ln 2 - 2 = 0.079.\]

Getting back to the question posed earlier, is it a good bet for you? Not really. You can expect to win about 8% of the time. With the
bet Linda is proposing, your odds of winning should be 1 to 4 (i.e.,
your probability of winning should equal 20%), since the money
you receive compared to what Linda receives is in the ratio of 4 to
1; 8% is far shy of 20%.

For the geometrical probability problems that follow, besides solving
them, you are also invited to construct variations of them (and then
solve these variations) and to look for real-world probability experi-
ments that they can model. And, to reverse the process, you are also
invited to think of a "mathematical probability experiment," and
then find a real-world situation that it models. An example of this is
the Place-Kicker Problem. In constructing this problem, we first
constructed the following problem so that specific mathematics (e.g.,
law of sines) had to be used in solving it.

Suppose we construct $\triangle ABC$ with side $AC$ of length $h$, the
length of side $AB$ chosen randomly between 0 and $2h$, and
angle $BAC$ chosen randomly between 0 and $\pi/2$. Find the
probability that angle $ABC$ is greater than $\pi/6$.

3.2 Exercises

1. Solve Example 2 with $x$ defined to be the distance between the
center of the square die face and the nearest parallel ruling (not
necessarily the one on the right.)

2. Find the probability that the roots of $x^2 + 2ax + b = 0$ are
complex, where $a, b \in [-B, B]$ and $B > 1$. Use your answer to
determine the probability that the roots are real. What is the
limiting probability that the roots are complex as $B \to \infty$?
Hence, what is the limiting probability that the roots are real as
$B \to \infty$?

3. Suppose we construct $\triangle ABC$ with $b = 1$, $c$ chosen randomly
between 0 and 1, and angle $A$ chosen randomly between 0 and
$\pi/2$. Find the probability that the area of $\triangle ABC$ is less than
$1/4$.

4. (Buffon Needle Problem) Let a needle of length $L$ be thrown at
random onto a board ruled with parallel lines spaced by a
distance $d$ from each other. What is the probability that the
needle will intersect one of these lines? Solve the problem for:
\begin{enumerate}
\item $d > L,$
\item $d < L.$
\end{enumerate}
(This is the oldest problem dealing with geometrical probability.
It was mentioned by George Louis Leclerc, Comte de Buffon)
(1707–1788), in the Proceedings of the Paris Academy of Science (1733) and later reproduced with its solution in Buffon's book Essai d'arithmétique morale, published in 1777.) Variations of this problem are given in Exercises 13–14 in Section 3.4.

5. Find the probability that the sum of two positive random numbers, each of which does not exceed 1, will not exceed 1, and their product will be at most 2/9.

6. State and solve a variation of Exercise 5.

7. (Variation of Example 2, Die Problem) Same problem statement except that now the parallel lines are at alternating distances of 4 cm and 8 cm, and the length of an edge of the die is 2 cm.

8. Let \( a \) and \( b \) be randomly chosen between 0 and 1. Construct \( \triangle ABC \) in the \( xy \)-plane with vertices \( A = (0,0) \), \( B = (a,0) \), and \( C = (1,b) \). Let \( M \) and \( N \) be the midpoints of sides \( AC \) and \( BC \), respectively, and let \( P \) be the point at which the line through \( M \) parallel to \( BC \) meets \( AB \). Find the probability that the area of quadrilateral \( PMNB \) is less than 1/16.

9. Use the answers to Example 2 and Variations 2B and 2C to predict an answer to this problem: A thin plate in the shape of a convex polygon, of dimensions so small that it cannot intersect two of the lines simultaneously, is thrown onto a board ruled as in the example and variations mentioned above. What is the probability that the boundary of the plate will intersect one of the lines?

10. A circle can be viewed as the limiting case of a regular polygon (i.e., as the number of sides of the regular polygon increases without bound). In light of your answer to Exercise 9, what do you predict for the probability that a circle will intersect one of the lines? Now determine the probability directly. (Note: You will have a one-dimensional sample space and event.)

### 3.3 Functions of Two Variables
(Double Integration)

If the region representing the sample space of the experiment is three-dimensional, it may be necessary to use multiple integrals to calculate volumes. The next two examples are of this type. The first
example is a variation of Example 2 (Die Problem). In this problem, a die is tossed onto a board with a rectangular grid, as opposed to the board ruled into a set of parallel lines in the Die Problem. Double integrals are used to calculate volumes; once the probability is found, it is easily seen that the Die Problem is a limiting case of this problem.

Example 6 (Die Problem with Rectangular Grid). Suppose a die is randomly tossed onto a grid composed of rectangles with dimensions \(a\) by \(b\) with both sides of the rectangle greater than a diagonal of a face of the die. Find the probability that the die will cross a grid line.

Solution. Let \(s\) be the length of an edge of the die. We have \(s\sqrt{2} < a\) and \(s\sqrt{2} < b\). Let \(x\) be the distance from the center of the die face to the nearest vertical line, and let \(y\) be the distance from the center of the die face to the nearest horizontal line. Let \(\theta\) be the angle that the die is rotated through so that each diagonal of the die is parallel to one of the sets of parallel lines forming the rectangular grid. These quantities are shown in Figure 14. Now, \(0 \leq x \leq a/2\), \(0 \leq y \leq b/2\), and \(-\pi/4 \leq \theta \leq \pi/4\). We have a success if \(x < (s\sqrt{2}/2)\cos \theta\) or \(y < (s\sqrt{2}/2)\cos \theta\). The sample space is the rectangular solid with dimensions \((a/2) \times (b/2) \times (\pi/2)\), and the success region is the union of two cylinders, as shown in Figure 15. The volume of the success region is

\[
\int_{-\pi/4}^{\pi/4} \int_{(s\sqrt{2}/2)\cos \theta}^{a/2} \frac{(s\sqrt{2}/2)\cos \theta}{s\sqrt{2}/2} \, dx \, d\theta
\]

\[
\quad \quad \quad \quad + \int_{-\pi/4}^{\pi/4} \int_{(s\sqrt{2}/2)\cos \theta}^{b/2} \frac{(s\sqrt{2}/2)\cos \theta}{s\sqrt{2}/2} \, dx \, d\theta,
\]

\[
= \frac{as}{2} - \left(\frac{s^2}{4}\right)(\frac{\pi}{2} + 1) + \frac{bs}{2}
\]

\[
= \frac{as}{2} + \frac{bs}{2} - \left(\frac{s^2}{4}\right)(\frac{\pi}{2} + 1).
\]

The volume of the sample space is \((\pi/2)(a/2)(b/2) = \pi ab/8\). Therefore, the probability of the die crossing a grid line is

\[
p = \frac{\frac{as}{2} + \frac{bs}{2} - \left(\frac{s^2}{4}\right)(\frac{\pi}{2} + 1)}{\pi ab/8}
\]

\[
= \frac{4as + 4bs - 2s^2(\frac{\pi}{2} + 1)}{\pi ab}
\]

\[
= \frac{4as + 4bs - s^2(\pi - 2)}{\pi ab}.
\]
Figure 14. Model for Example 6.

Figure 15. Sample space and success region for Example 6.
We observe that if we let \( b \to \infty \) with \( a \) fixed, then \( p \to 4s/\pi a \), which is the solution to the **Die Problem** with a set of parallel lines. (That this probability should approach the other is plausible. As the \( b \)-side of the rectangles increases without bound, the board resembles the **Die Problem** board.)

In **Example 5 (Acute Triangle Problem)**, a segment was randomly divided into three parts by two points chosen independently and randomly on the segment, and the probability was determined that the three parts are sides of an acute-angled triangle. We extend this example, by asking the same question but for a different hypothesis. We solve the problem by using a double integral to calculate the volume of the success region. The integration is quite easy; the challenge is setting up the integral. There is an alternate two-dimensional solution to the problem that does not require the use of calculus. It is one of the exercises appearing after this example.

**Example 7 (Acute Triangle on a Circle).** Three points \( A, B, \) and \( C \) are randomly chosen, independent of each other, on the circumference of a circle. What is the probability that all angles of triangle \( ABC \) are acute?

**Solution.** Choose a fixed radius of the circle. Then an outcome of the experiment can be represented by choosing angles \( \theta, \phi, \) and \( \beta \) such that \( 0 < \theta < \phi < \beta < 2\pi \), as shown in Figure 16. The

![Figure 16. Model for Example 7.](image)
sample space is
\[ S = \{(\theta, \phi, \beta) | 0 < \theta < \phi < \beta < 2\pi\} \]

and the success region is
\[ \{(\theta, \phi, \beta) \in S | \phi - \theta < \pi, \beta - \phi < \pi, \text{ and } 2\pi - \beta + \theta < \pi\} \].

These regions are graphed in Figure 17. The volume of the sample space is \((1/6)(2\pi)^3 = 4\pi^3/3\). The volume of the success region is
\[
\int_0^\pi \int_0^{2\pi} \left[ \theta + \pi - (\beta - \pi) \right] \, d\beta \, d\theta = \pi^3/3.
\]

Therefore
\[ \rho = \frac{\pi^3/3}{4\pi^3/3} = 1/4. \]

3.4 Exercises

11. Solve Example 7 (Acute Triangle on a Circle Problem) using two-dimensional regions. (Hint: Express the third central angle in terms of the other two.)
12. A line segment of unit length is divided into four parts by three points chosen at random on it.
   
   a. Find the probability that none of the four parts exceeds $a$ in length, where $1/2 \leq a \leq 1$.
   
   b. Also solve this problem if the line is divided into two parts by a point chosen at random on it.
   
   c. Finally, solve it if the line is divided into three parts by two points chosen at random on it.
   
   d. What would you predict the answer to be if the line is divided into five parts by four points chosen at random on it?

13. Variation of Buffon Needle Problem. Suppose a board is ruled by lines into congruent rectangles with sides of length $a$ and $b$. A needle of length $L$ ($L < a, L < b$) is thrown randomly onto the board.
   
   a. What is the probability that the needle will intersect a line?
   
   b. Use your answer to Exercise 13a to determine the probability for a grid consisting of squares of side $d$. (Note: Another interesting variation is for a grid of equilateral triangles whose sides are of length $a$, with the needle length $L$ less than the altitude of any of the triangles.)

14. In your answer to Exercise 13a, let $a \to \infty$. Compare your result with the answer to Exercise 4 (Buffon Needle Problem). Is what you see plausible?

15. Two afternoon maintenance workers in an automobile assembly plant have to make sure that the welding robots are working properly. Suppose that a particular robot broke down at some time between 2 and 2:10 P.M. and that each worker passes by that robot once at a random time during their working hours of 1:30 to 4:30 P.M.
   
   a. Find the probability that the robot broke down at a time between visits by the workers.
   
   b. Find the probability if the robot broke down precisely at 2 P.M.
c. Still assuming only one visit by each worker, how much should they extend their work hours past 4:30 P.M. so that the probability is $1/6$?

16. Construct and solve a “geometrical probability” problem whose solution requires finding a volume by double integration. Try to find a real-world application for which this is a model.

4. Simulation

An alternate method for deriving answers to geometrical probability problems is simulation. Two methods we present are real-world simulation and computer simulation.

4.1 Real-World Simulation

A real-world simulation of the experiment defined by Example 6 (Die Problem with Rectangular Grid) is an example of an experiment performed on a real-world model of the experiment. What you can do here is take a large piece of cardboard and rule it into congruent rectangles. Then randomly toss a square (the “die”), whose diagonal is smaller than either side of the rectangle, many times onto this surface and record the number of times you toss and the number of successes obtained. An empirical probability is determined by dividing the number of successes by the number of trials (i.e., the number of tosses). (The greater the number of tosses, the more faith you can place in your result.) You can compare this simulation result with the result determined analytically in Example 6 (you will have to substitute the length and width of your rectangle, and the length of a diagonal of your square, into the formula). A discussion with your classmates or instructor on which type of probability (empirical or analytical) you place more faith in may be lively.

4.2 Computer Simulation

A computer simulation of an experiment is an experiment performed on a mathematical model of the experiment. A computer program, using a random number generator, “performs” the experiment a specified number of times, determines which of the outcomes
are successes, and arrives at an empirical approximation of the probability we seek (by computing the ratio of the number of successes to the number of trials). (Looking at it from a geometrical standpoint, the probability—i.e., the ratio of the area (or volume) of the success region to the area (or volume) of the sample space—can be approximated by dividing the number of “computer-chosen” points that belong to the success region by the total number of “computer-chosen” points that belong to the sample space). In principle, computer simulation is no less exact than the analytical method, which also necessarily introduces an error, as the calculation must be stopped at a finite number of decimal places. In many cases, computer simulation is more advantageous, as it eliminates the need for setting up and evaluating (possibly difficult) integrals, which one must frequently do to obtain an analytical solution. Following are two BASIC programs, along with their outputs, which perform computer simulations of examples that were solved analytically (one by using single integrals and the other by using multiple integrals).

Example 8 (Acute Triangle Problem Simulation) (See Example 5; let the rod length be 1.)

10 REM ACUTE TRIANGLE PROBLEM
20 REM
30 REM VARIABLE NAMES:
40 REM N . . . NUMBER OF TRIALS OF THE EXPERIMENT
50 REM S . . . NUMBER OF SUCCESSES IN N TRIALS
60 REM I . . . COUNTER FOR FOR-NEXT LOOP
70 REM X . . . POSITION OF FIRST DIVISION POINT
80 REM Y . . . POSITION OF SECOND DIVISION POINT
100 REM TEMP . . . TEMPORARY STORAGE
110 REM LOCATION USED TO SWITCH THE VALUES OF X AND Y IF Y < X
120 REM
130 FOR N = 1000 TO 10000 STEP 1000
140 RANDOMIZE (N)
150 LET S = 0
160 FOR I = 1 TO N
170 LET X = RND
180 LET Y = RND
190 IF X <= Y GOTO 270
200 LET TEMP = Y
250 LET Y = X
260 LET X = TEMP
270 IF Y < 1/(2*(1-X)) AND Y > (2*(X**2) - 1)/(2*X - 1) AND X < 1/(2*Y) + Y - 1 THEN S = S + 1
280 NEXT I
290 PRINT "NUMBER OF TRIALS = "; N; " \ TAB (35);
295 " \ PROBABILITY = "; S/N
300 NEXT N

Program output:

NUMBER OF TRIALS = 1000 PROBABILITY = .097
NUMBER OF TRIALS = 2000 PROBABILITY = .075
NUMBER OF TRIALS = 3000 PROBABILITY = .077
NUMBER OF TRIALS = 4000 PROBABILITY = .0815
NUMBER OF TRIALS = 5000 PROBABILITY = .08519999
NUMBER OF TRIALS = 6000 PROBABILITY = .08066667
NUMBER OF TRIALS = 7000 PROBABILITY = .081
NUMBER OF TRIALS = 8000 PROBABILITY = .079
NUMBER OF TRIALS = 9000 PROBABILITY = .07988889
NUMBER OF TRIALS = 10000 PROBABILITY = .0799

Note that the analytical solution was 3 ln 2 - 2 = 0.079.

Example 9 (Die Problem with Rectangular Grid Simulation) (See Example 6; let a = 3, b = 2, s = 1.)

10REM DIE PROBLEM WITH RECTANGULAR GRID
20REM
30REM VARIABLE NAMES:
40REM PI . . . . THE CONSTANT PI
50REM N . . . . NUMBER OF TRIALS OF THE
60REM EXPERIMENT
70REM S . . . . NUMBER OF SUCCESSES IN N
80REM TRIALS
90REM I . . . . COUNTER FOR FOR-NEXT LOOP
100REM X . . . . DISTANCE FROM CENTER OF DIE
110REM FACE TO NEAREST VERTICAL
120REM LINE
130REM Y . . . . DISTANCE FROM CENTER OF DIE
140REM FACE TO NEAREST HORIZONTAL
150REM LINE
160REM THETA ANGLE THROUGH WHICH THE
170REM DIE FACE MUST BE ROTATED
180 REM SO THAT ONE DIAGONAL IS
190 REM VERTICAL AND THE OTHER IS
200 REM HORIZONTAL
210 REM
220 LET PI = 3.14159
230 FOR N = 1000 TO 10000 STEP 1000
240 RANDOMIZE (N)
250 LET S = 0
260 FOR I = 1 TO N
270 LET X = (3*RND)/2
280 LET Y = RND
290 LET THETA = (PI*RND)/2 - PI/4
300 IF X < SQRT(2)*COS(THETA)/2 OR
     Y < SQRT(2)*COS(THETA)/2 THEN
     S = S + 1
310 NEXT I
320 PRINT "NUMBER OF TRIALS = "; N; TAB (35);
     "PROBABILITY = "; S/N
330 NEXT N

Program output:
NUMBER OF TRIALS = 1000 PROBABILITY = .814
NUMBER OF TRIALS = 2000 PROBABILITY = .797
NUMBER OF TRIALS = 3000 PROBABILITY = .7776667
NUMBER OF TRIALS = 4000 PROBABILITY = .789
NUMBER OF TRIALS = 5000 PROBABILITY = .7858
NUMBER OF TRIALS = 6000 PROBABILITY = .784
NUMBER OF TRIALS = 7000 PROBABILITY = .79
NUMBER OF TRIALS = 8000 PROBABILITY = .783375
NUMBER OF TRIALS = 9000 PROBABILITY = .7864445
NUMBER OF TRIALS = 10000 PROBABILITY = .7915

Note that the analytical solution was

\[
\frac{4(3)(1) + 4(2)(1) - 1^2(\pi + 2)}{\pi(3)(2)} = 0.788.
\]

4.3 Exercises

Perform a computer simulation of the following problems. Use
10,000 trials. For each, compare your simulated result with the
mathematical result.

17. Example 4 (Dividing a Field Problem).
18. **Exercise 8 (Buffon Needle Problem)**. Use $L = 1$ and $d = 2$. Compare the reciprocal of the result with $\pi$, rounded to three decimal places.

19. **Example 7 (Acute Triangle on a Circle Problem)**.

## 5. Sample Exam

1. Find the probability that the roots of $x^2 + 2px + q = 0$ are real, for $a, b \in [-P, P]$ and $[Q, Q]$, respectively.

2. In $\triangle ABC$ suppose angle $A$ is $\pi/6$, $b = 2$, and $e = 3$. Two points $P$ and $Q$ are chosen at random on sides $AC$ and $AB$, respectively. Find the probability that quadrilateral $PQBC$ has area less than $3/4$.

3. If $x$, $y$, and $z$ are numbers selected at random in the interval $[0, a]$, find the probability that three line segments of lengths $x$, $y$, and $z$ can form a triangle.

4. Three points $A$, $B$, and $C$ are chosen at random on a line segment. Find the probability that $C$ lies between $A$ and $B$.

## 6. Answers to Exercises

**Variation 2A:** $(4e + 4e \cos \left((a/\sqrt{2}) - 4\sqrt{2} - a^2)/\pi a \right)$

**Variation 2B:** $3\pi/\pi a$.

**Variation 2C:** $(2m + 2\pi)/\pi a$.

**Variation 3A:** $1/2 - 3\pi^2/10A$.

**Variation 3B:** $(25 - \ln 72)/45$.

1. $4e/\pi a$.

2. $1/3\sqrt{R}, 1 - 1/3\sqrt{R}, 0, 1$.

3. $1/3 + (2/\pi)\ln\sqrt{2} + \sqrt{3}$.

4. a. $2L/\sigma d$.

   b. $2L(1 - \sin \theta)/\sigma d + 2\theta/\pi$ where $\cos \theta = d/L$.

5. $(3 + 2\ln 2)/9$.

6. $4/3\pi$.

7. $(1 + 2\ln 2)/4$.

8. circumference of the polygon/\pi d$.

9. circumference of the circle/\pi d$.
12. a. \(1 - 4(1 - a)^2\).
   b. \(2a - 1 = 1 - 2(1 - a)\).
   c. \(1 - 3(1 - a)^2\).
   d. \(1 - 5(1 - a)^4\).
13. a. \((2L(a + b) - L^2)/\pi ab\).
   b. \((4Ld - L^2)/\pi d\).
14. As \(a \to \infty\), you get the solution to Exercise 4. It is plausible since, as \(a \to \infty\), the grid of rectangles resembles a set of parallel lines (in a localized region).
15. a. 76/243
   b. 5/18
   c. 3.36 hours.

7. Answers to Sample Exam

1. \(1 - \sqrt{Q}/3P\) for \(P^2 > Q\), and \(1/2 + (1/6)(P^2/Q)\) for \(P^2 < Q\).
2. \((3 - 3 \ln 2)/6\).
3. 1/4.
4. 1/3.

About the Authors

Richard Dahlke began his mathematics teaching career in Wisconsin high schools. He was at SUNY-College at Buffalo prior to coming to the University of Michigan-Dearborn in 1967, where he is currently professor of mathematics. He was a visiting associate professor of mathematics at Oregon State University. He specializes in the mathematical preparation of teachers and is an active speaker and workshop leader. His research and writing interests in mathematics education focus on teaching methodology and student behavior. His previous publications include articles on geometrical probability. He received a B.S. at the University of Wisconsin (1980), an M.A.T. at California State University (1964), and an M.S. (1965) and a Ph.D. (1972) at the University of Michigan.

Robert Falker is an associate professor of mathematics at the University of Michigan-Dearborn, where he has been since 1972. He received his undergraduate degree at Mankato State University (1965) and both his master’s (1967) and doctoral degrees (1972) at the University of Minnesota. His previous mathematical publications include articles on group representations and geometrical probability. He teaches mathematics and computer science courses.