TILTUP PANELS:
LOCATE THE PULLEYS
by Gary Lee McGrath

The cable rigging for a tiltup panel.

APPLICATIONS OF CALCULUS TO ENGINEERING
Title: TILTUP PANELS: LOCATE THE PULLEYS

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Abstract: A current practice of the construction industry is to pour concrete panels on the ground. The panels are then tilted up to become part of the wall of a building. The stress analysis of tiltup panels indicates the need to locate the pulleys for different cable riggings. This module discusses the location of the pulleys for a cable rigging currently in use. The key differential equation is based on an explicit expression for the distance of a point from an ellipse. The Lagrange multiplier of the corresponding optimization problem appears explicitly in this expression and is computed numerically by Newton's method. The appearance of the Lagrange multiplier in the expression for the distance of a point from an ellipse and in the key differential equation is somewhat unusual.

Prerequisites: Intermediate calculus and Newton's method.
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1. INTRODUCTION

A few years ago, I was asked to carry out a stress analysis of concrete panels that were to be tilted up by a crane to become part of the wall of a gymnasium (Figure 1). Due to the special rigging of the cables, I needed to compute the distance of a point from an ellipse. Much to my surprise, I could not find an explicit solution of this problem in the literature. An application of Newton's method for two variables placated my interest for the moment. Unfortunately, I had devised and tested an algorithm for which it was difficult to prove convergence and which did not provide an explicit expression for the desired distance. Recently, I was able to derive an explicit expression for the distance of a point from an ellipse. This expression involves the Lagrange multiplier of the corresponding minimum distance problem. The Lagrange multiplier may be computed numerically using Newton's method. Having an explicit expression for the distance of a point from an ellipse makes it possible to obtain a differential equation for "locating the pulleys" of the cable rigging for a given angle of tilt. Once the pulleys are located, the stress analysis is a straightforward but tedious application of the Laws of Statics.

We will concentrate on the mathematics needed to compute the coordinates of each pulley for a given angle of tilt. As you shall see, the solution involves the use of Lagrange multipliers and Newton's method. What is peculiar to the tiltup problem, mathematically speaking, is that the numerical value of the Lagrange multiplier is essential and appears in the key differential equation.

It is possible to understand both the expression for the distance $d(w,E)$ of a point $w$ from an ellipse $E$ and how this expression enables us to derive the key differential equation used to locate the pulleys without ever computing such a distance or solving the key differential equation for a specific panel; but it is not advisable. The starred problems involve using a computer to apply the mathematical concepts to specific situations. Appendix 1 lists a well-tested BASIC program for computing $d(w,E)$. A program for locating the pulleys for a given angle of tilt is available upon request.
Figure 1. The cable rigging for a tiltup panel.

Scale:
1" = 10'

Heights of insert locations
\( F_1 = 27.9' \)
\( F_2 = 21.9' \)
\( F_3 = 16.2' \)
\( F_4 = 10.2' \)

Figure 2.
2. THE MODEL

The panels being currently fabricated admit considerable diversity (Figure 2). Mathematically, the panels result from "pasting together" trapezoids. Door and window openings may be created by "removing" trapezoids. Fortunately, the diversity of panel configurations does not affect the problem of locating the pulleys during a tiltup operation. Figure 2 illustrates a typical tiltup panel with a window opening. The dots represent the insert locations where the cable rigging is hooked to the panel.

A physical model of the tiltup operation can be easily constructed using thumb tacks, string and paper clips. The cable rigging for initial pickup and a 30° angle of tilt are shown in Figures 3 and 4 respectively.

The cable $l_1$ is fastened to the panel at points $F_1$ and $F_2$. The pulley $x$ traverses an ellipse
\[
\frac{(x_1 - p)^2}{a_1} + \frac{x_2^2}{a_2} = 1.
\]

Cable $l_2$ is fastened at points $F_3$ and $F_4$. The pulley $z$ likewise traverses an ellipse
\[
\frac{(z_1 - q)^2}{b_1} + \frac{z_2^2}{b_2} = 1.
\]

Finally the cable is attached to pulleys $x$ and $y$ and itself passes through the pulley $w$ as shown in Figure 1. The point $w$ travels along an elliptical path as the panel is tilted up (Figure 5). Our task is to locate the pulleys $w$, $x$, and $z$ for a given angle of tilt $\theta$. It is assumed that: 1) the crane operator raises the boom of the crane to keep the hoisting cable vertical; and 2) the cables are attached so that the panel tilts as shown in Figure 4. Figure 4 also illustrates the two sets of coordinate axes that we will use. The $e_1$, $e_2$ axes enable us to use the standard formulas for the ellipses traversed by the $x$ and $y$ pulleys. The $E_1$, $E_2$ axes provide "global" coordinates describing the tiltup operation as seen by an observer.
Figure 3. Pulley locations for initial pick up (0° angle of tilt).

Figure 4. Pulley locations for a 30° angle of tilt.
Exercise 1: Build a string model of Figure 3 where $F_1 = 3' \, F_2 = 7'$, $F_3 = 11'$, $F_4 = 13'$, $\xi_1 = 6'$, $\xi_2 = 4'$ and $\kappa = 12'$. Show that the equations for the two ellipses are given by

$$\left(\frac{x_1 - 5}{3}\right)^2 + \left(\frac{x_2}{\sqrt{5}}\right)^2 = 1 \quad \text{and} \quad \left(\frac{z_1 - 12}{2}\right)^2 + \left(\frac{z_2}{\sqrt{3}}\right)^2 = 1$$

respectively.

3. LAGRANGE MULTIPLIERS AND THE DISTANCE OF A POINT FROM A CURVE

The problem of finding the point $(x_1, x_2)$ on a curve $x_2 = f(x_1)$ that is nearest the point $(w_1, w_2)$ can be formulated as an optimization problem.

(1) Min $F = d^2 = (x_1 - w_1)^2 + (x_2 - w_2)^2$

subject to: $G(x_1, x_2) = x_2 - f(x_1) = 0$.

The Lagrangean for this problem is

(2) $L = F + \lambda G$,

where the number $\lambda$ is called the Lagrange multiplier. The point $(x_1, x_2)$ minimizing (1) is a critical point of the Lagrangean function (2); that is; the partial derivatives
\(L_{x_1} = L_{x_2} = 0\) at \((x_1, x_2)\).

**Example 1:** The distance of the point \((w_1, w_2)\) from a line \(a_1x_1 + a_2x_2 = d\). The corresponding optimization problem is

\[
\text{Min } F = D^2 = (x_1 - w_1)^2 + (x_2 - w_2)^2
\]

(3) subject to \(G = a_1x_1 + a_2x_2 - d = 0\).

The Lagrangean is

\[L = F + \lambda G = (x_1 - w_1)^2 + (x_2 - w_2)^2 + \lambda(a_1x_1 + a_2x_2 - d).\]

We compute the partial derivatives of \(L\) and set them equal to 0.

(4) \(L_{x_1} = 2(x_1 - w_1) + \lambda a_1 = 0\)

(5) \(L_{x_2} = 2(x_2 - w_2) + \lambda a_2 = 0\).

Multiplying (4) by \(a_2\) and (5) by \(a_1\) and eliminating \(\lambda a_1a_2\), we obtain

(6) \(a_2x_1 - a_1x_2 = a_2w_1 - a_1w_2\).

Multiplying (3) by \(a_2\) and (6) by \(a_1\) and subtracting yields

(7) \((a_1^2 + a_2^2)x_2 = a_2d - a_1a_2w_1 + a_1^2w_2\).

Similarly, multiplying (3) by \(a_1\) and (6) by \(a_2\) and adding gives

(8) \((a_1^2 + a_2^2)x_1 = a_1d + a_2^2w_1 - a_1a_2w_2\).

The point \((x_1, x_2)\) on the line \(a_1x_1 + a_2x_2 = d\) nearest to \((w_1, w_2)\) is thus

(9) \(x_1 = \frac{a_1d + a_2^2w_1 - a_1a_2w_2}{a_1^2 + a_2^2}\)

(10) \(x_2 = \frac{a_2d - a_1a_2w_1 + a_1^2w_2}{a_1^2 + a_2^2}\)

with the distance between \((x_1, x_2)\) and \((w_1, w_2)\) being
\[
D^2 = \frac{(a_1w_1 + a_2w_2 - d)^2}{a_1^2 + a_2^2}.
\]

You might have noticed that in solving Example 1, the Lagrange multiplier was eliminated and played no role in the final expressions for \(x_1, x_2\) or \(D\). This is typical of textbook problems. Try your hand on the following problems. Each problem has a unique constrained critical point. You may refer to reference [1] for a discussion of bordered Hessians and their use in classifying constrained critical points. Problems 2 - 6 will acquaint you with the use of Lagrange multipliers.

**Exercises**

2. Min \(D^2 = x_1^2 + x_2^2\) subject to \(x_1 + 2x_2 = 4\).

3. Min \(F = x_1x_2\) subject to \(x_1 + x_2 = 8\).

4. Min \(D^2 = (x_1 - 2)^2 + (x_2 - 2)^2\) subject to \(x_1 + x_2 = 1\).

5. Min \(F = x^2 + z^2\) subject to \(2x + 4z = 8\).

Interpret this problem in terms of a string of length \(\ell\).

6. Min \(F = 2x_1^2 + x_2^2\) subject to \(x_1 + x_2 = 1\).

7. Min \(F = \frac{4}{x_1} + x_1x_2 + \frac{8}{x_2}\) subject to \(x_1x_2 = 2\).

The problem of minimizing the distance of a point from an arbitrary curve is very difficult as you will soon discover if you venture forth from the established trails. In many instances numerical methods or just plain good luck are helpful. The following examples involve the solution of a cubic equation.
Example 2: The distance of a point \((w_1, w_2)\) from the parabola \(x_2 = x_1^2\).

\[
\begin{align*}
\min D^2 &= (x_1 - w_1)^2 + (x_2 - w_2)^2 \\
\text{subject to } x_2 &= x_1^2 \text{ where } w_2 \neq w_1^2.
\end{align*}
\]

We may proceed as in Example 1. Forming the Lagrangean

\[
L = (x_1 - w_1)^2 + (x_2 - w_2)^2 + \lambda(x_2 - x_1^2)
\]

and computing partials

\[
\begin{align*}
L_{x_1} &= 2(x_1 - w_1) - 2\lambda x_1 = 0 \\
L_{x_2} &= 2(x_2 - w_2) + \lambda = 0.
\end{align*}
\]

Eliminating \(\lambda\) yields

\[
\begin{align*}
x_1 - w_1 &= -2x_1(x_2 - w_2) = -2x(x_1^2 - w_2), \\
\text{or equivalently,}
2x_1^3 + (1 - 2w_2)x_1 - w_1 &= 0.
\end{align*}
\]

The cubic function in (15) has its point of inflection at \((0, -w_1)\). Equation (15) can also be obtained directly by substituting \(x_2 = x_1^2\) into the expression for \(D^2\) to obtain

\[
D^2 = (x_1 - w_1)^2 + (x_1^2 - w_2)^2.
\]

Equation (15) is \(1/2\) the derivative of expression (16) with respect to \(x_1\). If \(w_1 > 0\), the graph of \(D^2\) has one of the "W" shapes illustrated in Figure 6 and thus has at least one and at most three critical points. This raises the problem of classifying the critical points. The constrained critical points obtained by using Lagrange multipliers may be classified using Bordered Hessians. On the other hand, if we obtain equation (15) by working directly with expression (16), the second derivative test may be used to classify the critical points. From the geometry, we know that for \(w_1 > 0\), the desired critical point is the unique positive root of (15). Once this root \(x_1\) is calculated, the value of \(x_2\) is given by \(x_2 = x_1^2\) and the minimum distance \(D\) can be calculated from the standard distance formula.
Figure 6. The graph of $D^2$ with $\omega_1 > 0$.

Figure 7.

$(x_1, x_2) = (1/2, 1/4)$

$(\omega_1, \omega_2) = (3/2, -3/4)$
Example 3: Min $D^2 = (x_1 - 3/2)^2 + (x_2 + 3/4)^2$
subject to $x_2 = x_1^2$.

By (15), $x_1$ must satisfy

(17) $2x_1^3 + 2.5x_1 - 1.5 = 0$.

Thus $x_1 = 1/2$, $x_2 = 1/4$ and $D = \sqrt{2}$ (Figure 7).

Example 4: Min $D^2 = x_1^2 = x_1^2 + (x_2 - 4.5)^2$
subject to $x_2 = x_1^2$.

By (15) $x_1$ satisfies

(18) $2x_1^3 - 8x_1 = 0$.

Thus $(0,0)$, $(2,4)$ and $(-2,4)$ are possible solutions. The points on the parabola $x_2 = x_1^2$ nearest to $w_1 = 0$ and $w_2 = 4.5$ are $x_1 = \pm 2$, $x_2 = 4$ with $D = \sqrt{17}/2$ (Figure 8).

![Figure 8.](image)

Exercises

8. Min $D^2 = (x_1 - 3)^2 + x_2^2$
subject to $x_2 = x_1^2$.

9. Min $D^2 = (x_1 - \frac{4}{9})^2 + (x_2 - \frac{23}{18})^2$
subject to $x_2 = x_1^2$.
10. Min \( p^2 = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{4})^2 \)

subject to \( x_2 = x_1^2 \).

11. Min \( p^2 = (x - 1)^2 + (x - 2)^2 \)

subject to \( 4x_2 = x_1^2 \).

4. VECTORS

All vectors \( x = (x_1, x_2) \) will be two dimensional geometric vectors in the Cartesian plane. Let

\[
||x|| = \left[ x_1^2 + x_2^2 \right]^{1/2}
\]

denote the Euclidean norm or length of the vector \( x \), \( (x|z) = x_1z_1 + x_2z_2 \) the dot product of the vectors \( x \) and \( z \) and \( \dot{x} \) the derivative of the vector \( x \). Finally, if \( x = (x_1, x_2) \) we let \( x^\perp = (-x_2, x_1) \), called \( x \)-perp, be the vector obtained by rotating \( x \) through 90 degrees counterclockwise.

5. NEWTON'S METHOD

Newton's method is an iterative procedure for solving \( f(x^*) = 0 \). If we let \( x_0 \) be an initial guess for \( x^* \), the Newton iterates are

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.
\]

Assuming that: 1) \( x^* = \lim x_k \); 2) both \( f(x) \) and \( f'(x) \) are continuous at \( x^* \); and 3) \( f'(x^*) \neq 0 \), we see that

\[
\lim_{k \to \infty} x_{k+1} = \lim_{k \to \infty} x_k - \lim_{k \to \infty} \frac{f(x_k)}{f'(x_k)} = x^* - \frac{f(x^*)}{f'(x^*)}.
\]

Thus \( f(x^*) = 0 \) as desired.

Exercise 12. Write a computer program to approximate \( \sqrt{2} \) by applying Newton's method to the function \( f(x) = x^2 - 2 \). For which initial guesses \( x_0 > 0 \) do the Newton iterates converge?

An excellent discussion of Newton's method may be found in Atkinson's book [3]. Our application of Newton's method will be discussed in the next section.

-11-
6. THE DISTANCE OF A POINT FROM AN ELLIPSE

The problem is illustrated in Figure 9 and may be formulated as an optimization problem:

(19) \( \min D^2 = (x_1 - w_1)^2 + (x_2 - w_2)^2 \)

subject to \( \left( \frac{x_2 - p_1}{a_1} \right)^2 + \left( \frac{x_2 - p_2}{a_2} \right)^2 = 1 \)

where \( \left( \frac{w_1 - p_1}{a_1} \right)^2 + \left( \frac{w_2 - p_2}{a_2} \right)^2 > 1 \)

(\( w \) is outside the ellipse).

![Figure 9](image)

The solution is stated as a theorem with three parts.

THEOREM 1: The distance of a point from an ellipse.

Let \( \lambda \) be the Lagrange multiplier associated with the optimization problem (19) and define \( f(\lambda) \) as follows:

(20) \( f(\lambda) = \left( \frac{a_1(w_1 - p_1)}{\lambda + a_1^2} \right)^2 + \left( \frac{a_2(w_2 - p_2)}{\lambda + a_2^2} \right)^2 - 1. \)

Then:

1. (i) \( f(0) = \left( \frac{w_1 - p_1}{a_1} \right)^2 + \left( \frac{w_2 - p_2}{a_2} \right)^2 - 1 > 0. \)
(ii) \[ \lim_{\lambda \to -\infty} f(\lambda) = -1 \]

(iii) \( f \) is strictly decreasing on \([0, +\infty)\) since

\[
f'(\lambda) = -2 \left[ \frac{(a_1(w_1 - p_1))^2}{(\lambda + a_1^2)^3} + \frac{(a_2(w_2 - p_2))^2}{(\lambda + a_2^2)^3} \right] < 0.
\]

(iv) \( f \) is strictly convex on \([0, +\infty)\) since

\[
f''(\lambda) = 6 \left[ \frac{(a_1(w_1 - p_1))^2}{(\lambda + a_1^2)^4} + \frac{(a_2(w_2 - p_2))^2}{(\lambda + a_2^2)^4} \right] > 0.
\]

(v) The Lagrange multiplier \( \lambda \) is the unique positive root of \( f(\lambda) = 0 \) (Figure 10).

![Figure 10.](image)

(vi) If \( \lambda_0 = 0 \), the Newton iterates

\[
\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}
\]

increase monotonically with limit \( \lambda \).

2. The distance \( D \) of the point \( w = (w_1, w_2) \) from the ellipse is given by

\[
D^2 = \left[ \frac{(w_1 - p_1)}{a_1^2} \right]^2 + \left[ \frac{(w_2 - p_2)}{a_2^2} \right]^2.
\]

3. The point \( x = (x_1, x_2) \) on the ellipse nearest to \( w = (w_1, w_2) \) is
\[ x_1 = p_1 + \frac{w_1 - p_1}{1 + \frac{\lambda}{a_1^2}} \]

\[ x_2 = p_2 + \frac{w_2 - p_2}{1 + \frac{\lambda}{a_2^2}}. \]

Proof: First we will derive expression (20) using Lagrange multipliers. Forming the Lagrangean

\[ L = (x_1 - w_1)^2 + (x_2 - w_2)^2 + \]

\[ + \lambda \left[ \left( \frac{x_1 - p_1}{a_1^2} \right)^2 + \left( \frac{x_2 - p_2}{a_2^2} \right)^2 - 1 \right], \]

and computing partials yields

(21) \[ L_{x_1} = 2(x_1 - w_1) + 2\lambda \left( \frac{x_1 - p_1}{a_1^2} \right) = 0, \]

(22) \[ L_{x_2} = 2(x_2 - w_2) + 2\lambda \left( \frac{x_2 - p_2}{a_2^2} \right) = 0. \]

Expressions (21) and (22) imply

(23) \[ (1 + \frac{\lambda}{a_1^2})(x_1 - p_1) = w_1 - p_1 \]

and

(24) \[ (1 + \frac{\lambda}{a_2^2})(x_2 - p_2) = w_2 - p_2. \]

Substitute (23) and (24) into the equation of the ellipse to obtain the \( f(\lambda) = 0 \) of assertion (v) of Part 1.

Assertion 1.(i) through 1.(iv) of Part 1 follow by direct calculation. The Intermediate Value Theorem applied to \( f(\lambda) \) tells us that \( f(\lambda) = 0 \) has a unique positive root. We now claim that the Lagrange multiplier of equations (21) and (22) is the unique positive root of \( f(\lambda) = 0 \). Equations (21) and (22) may be written in vector form as \( \mathbf{w} - \mathbf{x} = \lambda \mathbf{G} \) where \( \mathbf{G} \) is the gradient of the constraint.
\[ g(x) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - 1 = 0. \]

Since \( g \) is a convex function, its gradient \( G \) is an outward normal at every point \( x \) satisfying \( g(x) = 0 \). Since \( w - x \) and \( G \) are parallel vectors in the same direction, we know that \( \lambda > 0 \). Thus assertion 1.(v) holds.

Assertion 1.(vi) simply states a well known fact on the monotonicity of Newton iterates \([2],[3]\).

Parts 2 and 3 follow directly from (23) and (24).

Q.E.D.

**Exercises.**

13*. Write a computer program to compute the unique positive root \( \lambda^* \) of \( f(\lambda) \) as defined in equation (20) where \( p = (0,0) \), \( a = (4,1) \) and \( w = (6,8) \).

*Hint:* Apply Newton's method with initial iterate \( \lambda_0 = 0 \).

14*. Use the value of \( \lambda^* \) obtained in Problem 13 to compute the distance of the point \((6,8)\) from the ellipse

\[ \left(\frac{x_1}{4}\right)^2 + x_2^2 = 1. \]

15*. Compare your answer for Problem 14* with the results obtained by running the program listed in Appendix I.

The program listed in Appendix I computes the distance \( D \) of a point \( w \) from the ellipse \( E \) where 1) \( E \) is centered at \((0,0)\), and 2) \( w \) is outside of \( E \). Newton's method is used to compute the unique positive root of \( f(\lambda) \). The program was tested by the following procedure:

(i) Values were assigned to \( a = (a_1,a_2) \) using uniformly distributed random numbers thus defining the ellipse

\[ E: \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1. \]

(ii) A random point \( x \) on the circumference of \( E \) was determined by selecting a uniformly distributed random number \( 0 \leq t \leq 2\pi \) and using the parametric equations of \( E \) to compute \( x \):
\[ x_1 = a_1 \cos t \]
\[ x_2 = a_2 \sin t. \]

(iii) Finally, the point \( w \) outside \( E \) was computed by picking a uniformly distributed random number \( D \) and moving from the point \( x \) exactly \( D \) units in the direction of the outward normal vector to \( E \) at \( x \). Thus \( d(w,E) = D \).

(iv) These values of \( a = (a_1, a_2) \) and \( w = (w_1, w_2) \) were then entered as data to the program of Appendix I. In this way, the computed value \( D \) of \( d(w,E) \) could be compared with the exact value \( D \). The results of these tests are given in Table 1. The programs were run on the Prime 750 using double precision BASIC with 14 significant digits.

<table>
<thead>
<tr>
<th>Test #</th>
<th># of test runs</th>
<th>range of ( a_1, a_2 )</th>
<th>range of ( D )</th>
<th>criteria for success</th>
<th>% of successes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,000</td>
<td>0.001 to 1000</td>
<td>0 + 1000</td>
<td>(</td>
<td>\bar{D} - D</td>
</tr>
<tr>
<td>2</td>
<td>1,000</td>
<td>0.001 to 10,000</td>
<td>0 + 10,000</td>
<td>(</td>
<td>\bar{D} - D</td>
</tr>
<tr>
<td>3</td>
<td>10,000</td>
<td>0.000001 to 100,000</td>
<td>0 + 100,000</td>
<td>(</td>
<td>\bar{D} - D</td>
</tr>
</tbody>
</table>

7. THE KEY DIFFERENTIAL EQUATION

We wish to locate the top pulley \( w \) (as shown in Figure 11) where the crane is lifting the panel. We recall from the proof of assertion 1.(v) of Theorem 1 that to locate this point, we may follow the normal vector \( N = T^\perp \) a distance \( c||N|| \). To compute this distance, we need to determine the value of the scalar \( c \). This is done by deriving the key differential equation for \( c \) and solving this equation numerically.

First, we parametrize the first ellipse of Figures 3 and 11 by
(25) \[ x_1 = p - a_1 \cos(t) \]
\[ x_2 = a_2 \sin(t) \text{ for } 0 \leq t \leq \pi. \]

The Tangent vector \( T \) and the outward Normal vector \( N \) to the first ellipse centered at \( p \) are:

(26) \[ T_1 = x_1 = a_1 \sin(t) \]
\[ T_2 = x_2 = a_2 \cos(t) \]
and
\[ N_1 = -a_2 \cos(t) \]
\[ N_2 = a_1 \sin(t) \]
respectively.

Let \( c \) be the scalar defined by the vector equation

(27) \[ w = x + cN. \]

Once again direct calculation gives

(28) \[ \dot{w} = \dot{x} + c\dot{N} + \dot{c}N = \dot{T} + cM + \dot{c}N \]
where \( M = \dot{N} \) is given by

\[ M_1 = \dot{N}_1 = a_2 \sin(t) \]
\[ M_2 = \dot{N}_2 = a_1 \cos(t). \]

![Diagram](image.png)

**Figure 11.**
We now state the key differential equation for c.

**THEOREM 2:** The Key Differential Equation.

The scalar c (Figure 11) satisfies the differential equation

$$\dot{c} \left||N|| + \frac{(G \cdot N)}{||N||} \right| = -c \left[ \left|\frac{(N \cdot M)}{||N||} + \frac{(G \cdot M)}{||G||} \right| - \frac{(G \cdot T)}{||G||} \right]$$

where the vector G is given by

$$G = w - z = (G_1, G_2)$$

(30) \[ G_1 = \frac{w_1 - q}{1 + b_1^2 \mu} \quad \text{and} \quad G_2 = \frac{w_2 - b_2}{1 + b_2^2 \mu} \]

The initial condition c(0) is given by

(31) \[ c(0) = \frac{(w_1(0) - x_1(0))}{N_1(0)} \]

where

$$w_1(0) = \frac{((p - a_1) + (q - b_1))}{2} $$

$$x_1(0) = p - a_1 \quad \text{and}$$

$$N_1(0) = -a_2.$$

Of course, u is the Lagrange multiplier associated with the distance of the point w from Ellipse 2 as shown in Figure 11.

**Proof of THEOREM 2:** Expression (30) for the vector G is an immediate consequence of THEOREM 1, Part 3. We recall that according to the proof of assertion 1.(v) of THEOREM 1, the vector G is perpendicular to the second ellipse centered at q.

We derive (29) by differentiating

$$||w - x|| + ||w - z|| = \lambda$$

or equivalently

(32) \[ c \left|\frac{N}{||N||} + ||w - z|| \right| = \lambda. \]

The derivation proceeds in five steps.

**Step 1.** \[ ||N|| = \left[ N_1^2 + N_2^2 \right]^{1/2} \text{ with} \]

-18-
\[ (33) \frac{d}{dt} || \mathbf{N} || = \left[ N_1 \dot{N}_1 + \frac{N_2 \dot{N}_2}{N_1 + N_2} \right] \frac{1}{2} = \frac{\{N \dot{N}\}}{||N||} = \frac{\{N \dot{M}\}}{||N||} \]

Step 2: By Theorem 1,

\[ (34) \quad ||w - z|| = \left( \frac{w_1 - \frac{q}{b_1}}{1 + \frac{b_1}{\mu}} \right)^2 + \left( \frac{w_2 - \frac{b_2^2}{b_2}}{1 + \frac{b_2}{\mu}} \right)^2 \]

Thus \( \frac{d}{dt} ||w - z|| \) can be computed to obtain

\[ (35) \quad \frac{d}{dt} ||w - z|| = \frac{1}{||w - z||} \left[ \frac{w_1 - \frac{q}{b_1}}{1 + \frac{b_1}{\mu}} \frac{\dot{w}_1}{\mu} + \frac{w_2 - \frac{b_2^2}{b_2}}{1 + \frac{b_2}{\mu}} \frac{\dot{w}_2}{\mu} \right. \\
+ \left. \frac{b_1^2 (w_1 - q)^2}{(1 + b_1^3)^3} + \frac{b_2^2 w_2^2}{(1 + \frac{b_2^3}{\mu})^3} \right] \]

Step 3: To calculate the derivative \( \dot{\mu} \) of the Lagrange multiplier \( \mu \), we recall from Theorem 1 that \( \mu \) is the root of

\[ (36) \quad g(\mu) = \left( \frac{b_1 (w_1 - q)}{\mu + b_1^2} \right)^2 + \left( \frac{b_2 w_2}{\mu + b_2^2} \right)^2 - 1 = 0. \]

Differentiating (36), we obtain

\[ (37) \quad \frac{\dot{\mu}}{\mu} \left[ \frac{b_1^2 (w_1 - q)^2}{(\mu + b_1^2)^3} + \frac{b_2^2 w_2^2}{(\mu + b_2^2)^3} \right] = \frac{b_1^2 (w_1 - q) \dot{w}_1}{(\mu + b_1^2)^2} + \frac{b_2^2 w_2 \dot{w}_2}{(\mu + b_2^2)^2}. \]

Factoring \( \mu \) out of the denominator of (37) gives

\[ (38) \quad \frac{\dot{\mu}}{\mu} \left[ \frac{b_1^2 (w_1 - q)^2}{1 + \frac{b_1^2}{\mu}} + \frac{b_2^2 w_2^2}{1 + \frac{b_2^2}{\mu}} \right] = \frac{b_1^2 (w_1 - q) \dot{w}_1}{1 + \frac{b_1^2}{\mu}} + \frac{b_2^2 w_2 \dot{w}_2}{1 + \frac{b_2^2}{\mu}}. \]
Step 4: Substitute (38) and (35) to obtain
\[
\frac{d}{dt} ||w - z|| = \frac{|G \dot{w}|}{||G||}.
\]

Step 5: We use steps 1 - 4 to obtain the key differential equation
\[
||w - x|| + ||w - z|| = \ell,
\]
and thus
\[
c ||K|| + \left[ \frac{\left( \frac{w_1 - q}{b_1^2} \right)^2}{1 + \frac{1}{\mu}} + \frac{\left( \frac{w_2}{b_2^2} \right)^2}{1 + \frac{1}{\mu}} \right]^{1/2} = \ell.
\]

Differentiating and using the results of steps 1 - 4, we have
\[
\dot{c} ||N|| + c \frac{(N \dot{N})}{||N||} + \frac{|G \dot{w}|}{||G||} = 0,
\]
where \( \dot{w} = T + cM + \dot{\ell}N \). Substituting for \( \dot{w} \) yields
\[
\dot{c} ||N|| + c \frac{(N \dot{N})}{||N||} + \frac{|G (T + cM + \dot{\ell}N)|}{||G||} = 0.
\]
Finally, rearranging terms, we have
\[
(39) \quad \dot{c} \left( ||N|| + \frac{|G|}{||G||} \right) = -c \left( \frac{(N \dot{N})}{||N||} + \frac{|G|}{||G||} \right) - \frac{|G|}{||G||}.
\]

O.B.D.

The differential equation for \( c \) can be solved numerically by a Runge Kutta method of order 2. Let
\[
\dot{c}(t) = F(t, c) \quad \text{where} \quad F \text{ is defined by}
\]
\[
(40) \quad F(t, c) = -\left( \frac{(N \dot{N})}{||N||} + \frac{|G|}{||G||} \right) - \frac{|G|}{||G||} \cdot c - \frac{(G \dot{T})}{||G||} \cdot \frac{||N||}{||G||} - \frac{(G \dot{M})}{||G||} \cdot \frac{||N||}{||G||}.
\]

Then a Runge Kutta approximate of order 2 for \( c(t + h) \) is given by
\[
(41) \quad c(t + h) = c(t) + h F(t, c(t)) + F(t + h, c(t) + hc(t)) \frac{h}{2}.
\]

For known values of \( t \) and \( c(t) \), the evaluation of \( F(t, c(t)) \) involves the calculation of \( w(t) = x(t) + c(t)N(t) \) and the computation of \( G(t) \). The vector \( G(t) \) is computed in three
steps: first, \( w(t) \) is obtained by using Newton's method to solve the equation

\[
(42) \quad f(u(t)) = \left( \frac{b_1(w_1(t) - q)}{\mu + b_1^2} \right)^2 + \left( \frac{a_2 w_2(t)}{\mu + b_2^2} \right)^2 - 1 = 0;
\]

second, by THEOREM 1, Step 3, \( z(t) \) is given by

\[
(43) \quad z_1(t) = q + \frac{w_1(t) - q}{1 + \frac{\mu(t)}{b_1^2}} \quad \text{and} \quad z_2(t) = \frac{w_2(t)}{1 + \frac{\mu(t)}{b_2^2}};
\]

finally, \( G(t) = w(t) - z(t) \). Consequently, each Runge Kutta iteration involves two applications of Newton's method.

The initial condition for \( c(0) \) is computed as stated in THEOREM 2. Given the angle of tilt \( \Theta \) (Figure 4), the Runge Kutta iterations proceed until the argument \( \Theta_0 \) of the outward normal vector \( \mathbf{w}^\perp \) of the path traced by \( \mathbf{w}(t) \) (Figure 5) is equal to \( 90 - \Theta \); the argument \( \Theta_0 \) of \( \mathbf{w}^\perp \) is computed relative to the \( e_1 \) and \( e_2 \) axes (Figures 4 and 11).

Table 2 locates the pulleys \( x, w \) and \( z \) for the specific tiltup panel of Figures 2, 3 and 4. The panel is 35 feet high and the foci \( F_1, F_2, F_3 \) and \( F_4 \) are 10.2', 16.2', 21.9' and 27.9' respectively. The first ellipse is centered at

\[
p = \frac{F_1 + F_2}{2} = 13.2';
\]

the cable has a length of \( k_1 = 20' \). Consequently,

\[
a_1 = \frac{k_1}{2} = 10 \quad \text{and} \quad a_2 = \left[ \left( \frac{k_1}{2} \right)^2 - \left( \frac{F_2 - F_1}{2} \right)^2 \right]^{1/2} = \sqrt{91}.
\]

The second ellipse is centered at

\[
q = \frac{F_3 + F_4}{2} = 24.9';
\]

the cable has a length of \( k_2 = 25' \). Hence

\[
b_1 = \frac{L_2}{2} = 12.5'\]

and
\[ b_2 = \left( \frac{f_2}{2} \right)^2 - \left( \frac{F_4 - F_2}{2} \right)^2 \right)^{1/2} = \sqrt{589}. \]

The top cable \( i \) has a length of 25'. All coordinates in Table 2 are given relative to the \( E_1, E_2 \) axes of Figure 4 and thus describe the tiltup operation as seen by an observer.

<table>
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<th>( \theta ) (°)</th>
<th>( w_1 )</th>
<th>( x_1 )</th>
<th>( z_1 )</th>
<th>( w_2 )</th>
<th>( x_2 )</th>
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<td>15.63</td>
<td>21.59</td>
<td>25.95</td>
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<td>37.00</td>
</tr>
</tbody>
</table>

-22-
Exercise 16. Derive the key differential equation for the rigging of Figure 12.

Project: Write a computer program to verify Table 2.

Figure 12.

8. REFERENCES


4. See also *The Distance from a Point to an Ellipsoid*, Mathematics Magazine (to appear).

9. SOLUTIONS

2. \( x_1 = \frac{4}{5}, x_2 = 8.5 \) and Min = 16/5.

3. \( x_1 = 4, x_2 = 4 \) and Max = 16.

4. \( x_1 = 6/5, x_2 = 8/5 \) and Min = 4/5.

5. \( r = \frac{\sqrt{2}}{2(4 + \pi)} \), \( z = \frac{\sqrt{2}}{4 + \pi} \) and Min = \( \frac{\sqrt{2}}{4(4 + \pi)} \).

6. \( x_1 = 1/3, x_2 = 2/3 \) and Min = 2/3.

7. \( x_1 = 1, x_2 = 2 \) and Min = 10.

8. \( x_1 = 1, x_2 = 1 \) and Min = 5.
APPENDIX 1

The following program computes the distance between a point P and an ellipsoid E centered at the origin in Euclidean N-space. The axes of the ellipsoid lie along the coordinate axes.

20 PRINT "WELCOME TO TILTUP-APPENDIX1"
30 PRINT
40
60 MIN // X - P //**2
70
80 XT*Q*X = 1 WHERE 1. Q=DIAG(1/A(1)**2,...,
1/A(N)**2) > 0
100 2. PT*Q*P > 1
120 (THE POINT P IS OUTSIDE
THE ELLIPSOID.)
140
160 THIS PROGRAM COMPUTES THE DISTANCE D(P,E) OF THE
POINT P
170
180 FROM THE ELLIPSOID XT*Q*X = 1 DEFINED ABOVE.
185
190 THE POINT P IS ASSUMED TO BE OUTSIDE THE
ELLIPSOID.
195
200 DIM A(10),P(10),X(10)
220
240 PRINT "INPUT N THE DIMENSION OF THE ELLIPSOID"
260 INPUT N
280 PRINT "INPUT THE A VECTOR"
300 FOR I=1 TO N
320 INPUT A(I)
340 NEXT I
360 PRINT "INPUT P THE POINT OUTSIDE THE ELLIPSOID"
380 FOR I=1 TO N
400 INPUT P(I)
420 NEXT I
440 CHECK TO SEE IF THE POINT P IS REALLY OUTSIDE
THE ELLIPSOID
424 F=0
426 FOR I=1 TO N
428 F=F (P(I)/A(I))**2
430 NEXT I
432 P=P-1
434 IF P>0 THEN 520
436 PRINT "THE POINT P IS NOT OUTSIDE THE ELLIPSOID"
438 GO TO 99999
440
-25-
USE NEWTON'S METHOD TO FIND THE UNIQUE POSITIVE ROOT OF:

\[ F(L) = \frac{(A(1)\cdot p(1))/(L+A(1)^2+\ldots+(A(N)\cdot p(N))/(L+A(N)^2)) - 1}{0} \]

520 PRINT "DO YOU WISH ALL NEWTON ITERATES PRINTED? (YES/NO)"
530 INPUT AS
540 IF A$="NO" THEN 565
550 Q=0
560 GO TO 600
565 IF A$="YES" THEN 580
570 Q=1
575 GO TO 600
580 PRINT "PLEASE RE-ENTER YOUR ANSWER: YES/NO"
585 GO TO 520
600 IF Q=0 THEN 640
605 PRINT
610 PRINT "NEWTON'S METHOD"
615 PRINT
620 PRINT "STOPLING CRITERION: ABS(L(N+1)-L(N))<E"
630 PRINT "E=0.000001"
640 E=0.000001
650 PRINT
660 IF Q=0 THEN 690
670 PRINT "ITERATION #, NEWTON ITERATES: L(N), L(N+1),
AND E=L(N+1)-L(N)"
690 PRINT
695 M=1
700 L=0 THE INITIAL NEWTON ITERATE
740
760 F=0 CALCULATE F(L)
780 FOR I=1 TO N
800 F=F+((A(I)\cdot p(I))/(L+A(I)^2))**2
820 NEXT I
840 E=F-L
860 D=0 CALCULATE D(L) THE DERIVATIVE OF F(L)
880 FOR I=1 TO N
900 D=D+((A(I)\cdot p(I)^2)/(L+A(I)^2))**3
920 NEXT I
940 D=-2*D
960
980 T=-F/D
990 L1=L+T
1000
1040 IF Q=0 THEN 1140
1060 PRINT M;":";TAB(5);L;TAB(25);L1;TAB(45);T
1120
1140 IF ABS(T)<E THEN 1190
1150
1160 L=L1
1165 M=M+1
1170 GO TO 760
1180
1190 PRINT
1200 PRINT"EXIT NEWTON"
1210
1220 CALCULATE X
1230 FOR I=1 TO N
1240 X(I)=((A(I)**2)*P(I))/(L+A(I)**2)
1250 NEXT I
1260 CALCULATE THE LENGTH OF X-P
1270 S=0
1280 FOR I=1 TO N
1290 S=S+(X(I)-P(I))**2
1300 NEXT I
1310 S=SQR(S)
1320
1440 PRINT
1450 PRINT"THE MINIMUM DISTANCE OF P FROM THE ELLIPSOID: " ; S
1460 PRINT
1470 PRINT"THE LAGRANGE MULTIPLIER L = " ; L
1480 PRINT"THE POINT X ON THE ELLIPSOID CLOSEST TO P"
1490 FOR I=1 TO N
1500 PRINT I," " ; X(I)
1510 NEXT I
1520
99999 END