Low influence functions over slices of the Boolean hypercube depend on few coordinates

Karl Wimmer
Duquesne University
wimmerk@duq.edu

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Abstract

One of the classic results in analysis of Boolean functions is a result of Friedgut [Fri98] that states that Boolean functions over the hypercube of low influence are essentially determined by few coordinates, called juntas. While this result has also been extended to product distributions, not much is known in the case of nonproduct distributions.

We generalize this result to slices of the Boolean cube. A slice of the Boolean cube is the set of strings with some fixed Hamming weight. In this setting, we define the notion of influence and determine a natural orthogonal basis for functions over these domains. We essentially follow the proof for the uniform distribution case, but the setup in order to do so is highly nontrivial.

The main techniques used are combinatorics of Young tableaux motivated by the representation theory of the symmetric group along with an application of hypercontractivity in slices of the Boolean hypercube due to O’Donnell and Wimmer [OW09].

1 Introduction

1.1 Background

In characterizing properties of Boolean functions, the influence of a Boolean function is a very natural complexity measure. For a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, we define

$$\text{Inf}_i(f) = \Pr_{x}[f(x) \neq f(x^{(i)})] \quad \text{and} \quad \text{Inf}(f) = \sum_{i=1}^{n} \text{Inf}_i(f),$$

where $x^{(i)}$ denotes $x$ with the $i$th bit flipped, and the probability is with respect to the uniform distribution\(^1\). We say that $\text{Inf}_i(f)$ is the influence of the $i$th coordinate (on $f$), and that $\text{Inf}(f)$ is the (total) influence of $f$.

One classical stability result in the analysis of Boolean functions is Friedgut’s junta theorem [Fri98]. This result states the following:

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\(^1\)All probabilities and expectations contained in this manuscript are with respect to the uniform distribution.
Theorem 1.1. For any \( f : \{0,1\}^n \to \{0,1\} \), there is a function \( g : \{0,1\}^n \to \{0,1\} \) such that \( g \) depends on at most \( 2^{O(\inf(f)/\epsilon)} \) coordinates and \( \Pr_x[f(x) \neq g(x)] \leq \epsilon \).

This theorem has seen many applications in such areas as hardness of approximation and CSPs [DS04, CKK+06, KR08] and computational learning theory [OS08]. One of the original applications for Friedgut’s junta theorem was understanding threshold phenomena in connection with the Russo-Margulis Lemma [Mar74, Rus82].

A celebrated theorem in a similar spirit is the Kahn-Kalai-Linial (KKL) Theorem [KKL88]:

Theorem 1.2. For any \( f : \{0,1\}^n \to \{0,1\} \), there is a coordinate \( i \in [n] \) such that \( \inf_i(f) \geq \var(f)\Omega(\frac{\log n}{n}) \).

Recently, O’Donnell and the author [OW09] proved a broad generalization of the KKL Theorem. This generalization applies to Boolean functions over any domain that enjoys a certain kind of hypercontractivity, which roughly says that small sets “expand”. In the same paper, this theorem is used to prove a stable version of the classical Kruskal-Katona Theorem. This Kruskal-Katona Theorem is in turn used to prove a conjecture of Blum, Burch, and Langford [BBL98] concerning learning monotone functions.

Part of what makes a succinct generalized KKL Theorem proof possible is the fact that the properties \( \inf_i(f) \) and \( \var(f) \) have straightforward generalizations to other spaces. In contrast, Friedgut’s junta theorem is more difficult, because it is not clear how to even define a junta in general spaces. In the case that the domain has a product structure, a natural candidate is to simply define a junta over the components of the structure. We note that product distribution versions of Friedgut’s junta theorem are known; Friedgut himself proved such a result in his original paper [Fri98]. Friedgut considered the \( p \)-biased case, where all the coordinates have the same marginal distribution. Recently, Sachdeva and Tulsiani [ST11] proved a generalization of Friedgut’s junta theorem to Cartesian products of graphs. The related work of Hatami [Hat] proves a similar in spirit statement for a broader class of functions called pseudo-juntas; this work is also focused on product distributions.

However, for such a theorem to hold in nonproduct settings, it seems necessary to work in a very specific scenario. Perhaps the most interesting and simplest nontrivial case to consider is functions of the form \( g : \binom{[n]}{k} \to \{0,1\} \); these are functions over all binary strings of length \( n \) with Hamming weight \( k \). This scenario is the same as the one focused on in [OW09] in order to prove a stable version of the Kruskal-Katona Theorem. Their analysis uses the Schreier graph of the symmetric group \( S_n \) acting on \( \binom{[n]}{k} \), where \( S_n \) is generated by the \( \binom{n}{2} \) transpositions.

It is not immediately clear what a junta should even be in the case of \( f : S_n \to \{0,1\} \), although the work of Ellis and others [EFFa, EFFb, EFFc, Ell11] discusses stability results and Boolean functions of this form. Our techniques are similar to the ones found there. We do not directly use any extremal combinatorics here, although the result can be viewed as a statement in that setting.

1.2 Our results

Our main result is the following.

Theorem 1.3. Let \( g : \binom{[n]}{k} \to \{0,1\} \) be such that \( \inf(f) \leq \ell n \) and \( k/n \) is bounded away from 0 and 1. For every \( \epsilon > 0 \), there exists a function \( h : \binom{[n]}{k} \to \{0,1\} \) such that \( h \) depends only on \( \exp(O(\ell/\epsilon)) \) many coordinates and \( \Pr_x[g(x) \neq h(x)] \leq \epsilon \).

\(^2\)We will suppress most of the details of the Schreier graph analysis found in [OW09] in the main narrative, but we include the details in Appendix A.
Our definition of influence in the above theorem uses

\[ \text{Inf}_{ij}(f) = \Pr_x[f(x) \neq f(x_{ij})] \quad \text{and} \quad \text{Inf}(f) = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}(f), \]

where the write \( x_{ij} \) to mean \( x \) with the \( i \)th and \( j \)th bits transposed. We remark that here, we will consider total influence here rather than (twice the) average influence \( \mathcal{E}[f] \) as in [OW09].

We note that the influence condition in the above theorem is natural. For example, the function \( g : \binom{[n]}{k} \to \{-1, 1\} \) such that \( g(x) = x_1 \) has \( \text{Inf}_{1j}(g) = \frac{2k(n-k)}{n(n-1)} \) if \( j = 2, 3, \ldots, n \) and \( \text{Inf}_{ij}(g) = 0 \) if \( i \neq 1 \) and \( j \neq 1 \). Thus \( \text{Inf}(g) = \frac{2k(n-k)}{n-1} \), which is at most \( \frac{n^2}{2(n-1)} \sim \frac{n}{2} \), occurring when \( k = n/2 \).

The primary technique used is the representation theory of the symmetric group. We analyze Young’s Orthogonal Representation for the symmetric group, and “restrict” ourselves to functions that are consistent with some \( g : \binom{[n]}{k} \to \{0, 1\} \). Specifically, given \( g : \binom{[n]}{k} \to \mathbb{R} \), we construct \( f^g : S_n \to \mathbb{R} \) by associating each string in \( \binom{[n]}{k} \) with \( kl(n-k)! \) many permutations in a natural way. We focus our attention on analyzing the structure of \( f^g \), then argue that this analysis can be projected back down to \( g \) in a meaningful way.

Ultimately, we will proceed similar to the proof of Friedgut’s theorem presented in [O’D07]. We begin by setting up the necessary Fourier analysis. We also analyze the interplay between the basis from Fourier analysis and the basis yielded by studying the Cayley graph of \( S_n \) generated by transpositions. This analysis will allow us to apply hypercontractivity as in [OW09].

The bulk of our work entails deriving a Fourier interpretation for the effect of the non-influential coordinates. Armed with this expression, we show that under a further assumption related to hypercontractivity and the function’s influences (which is always satisfied when we project down to the domain \( \binom{[n]}{k} \)), the contribution to the Fourier spectrum is very little from the non-influential coordinates. We conclude that our original function essentially depends on few coordinates.

In [OW13] it is shown that the generalized KKL Theorem can not be improved for functions \( f : S_n \to \{0, 1\} \). We show a similar statement for analogues of Friedgut’s junta theorem.

**Theorem 1.4.** There exists a function \( f : S_n \to \{0, 1\} \) such that \( \text{Inf}(f) \leq 2n \) but any function \( h : S_n \to \{0, 1\} \) where \( \Pr_{\sigma}[f(\sigma) \neq h(\sigma)] \leq 1/4 \) depends on at least \( n/12 \) coordinates.

The above theorem uses

\[ \text{Inf}_{ij}(f) = \Pr_{\sigma}[f(\sigma) \neq f(\sigma_{ij})] \quad \text{and} \quad \text{Inf}(f) = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}(f), \]

and \( h \) does not depend on coordinates in \( S \subseteq [n] \) if \( \text{Inf}_{ij}(h) = 0 \) for all \( i, j \in S \). We prove this result in Appendix C.

### 1.3 Outline

In Section 2, we provide a short primer on representation theory in the general case, then we provide the basics of representation theory of the symmetric group in Section 3. In Section 4, we derive expressions for influences in the symmetric group. We do most of the heavy work that is quite similar to the proof of Friedgut’s junta theorem in [O’D07] in Section 5. We finally prove the main theorem in Section 6.
2 Representation theory for general groups

We begin with some general representation theory.

**Definition 2.1.** We say that a representation \( \rho \) of a group \( G \) is a mapping \( \rho : G \to \mathbb{R}^{d_\rho \times d_\rho} \) which preserves the algebraic structure of \( G \); that is, for \( \sigma_1, \sigma_2 \in G \) we have \( \rho(\sigma_1 \sigma_2) = \rho(\sigma_1) \cdot \rho(\sigma_2) \). The matrices in the codomain of \( \rho \) are called the representation matrices, and \( d_\rho \) is the degree of the representation.

We will be concerned with representations that are **irreducible**. These are representations that cannot be decomposed into simpler representations (for some suitable definition of decompose). A set of all possible irreducible representations \( \mathcal{R} \) contains all information about the structure of \( G \).

The Peter-Weyl Theorem says that the functions given in the matrix entries of irreducible representations of \( G \) form an orthogonal basis for \( L^2(G) \). For any group \( G \), we have the following definition.

**Definition 2.2.** Let \( f : G \to \mathbb{R} \) be any function on \( G \), and let \( \rho \) be any representation on \( G \). The Fourier coefficient of \( f \) at the representation \( \rho \) is given by the matrix

\[
\hat{f}_\rho = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \rho(\sigma)
\]

The collection of the matrices \( \hat{f}_\rho \) at irreducible representations of \( G \) is called the Fourier transform of \( f \).

We can reconstruct \( f \) from its Fourier transform via the Fourier Inversion formula:

\[
f(\sigma) = \sum_{\rho \in \mathcal{R}} d_\rho \text{tr} \left( \hat{f}_\rho^t \rho(\sigma) \right)
\]

using the notation \( A^t \) for the transpose of a matrix \( A \). The \( \text{tr} \) expression can be thought of as a dot product between two length-\( d_\rho^2 \) vector versions of \( \hat{f}_\rho \) and \( \rho \), arranging each matrix into a vector by taking the elements first top to bottom, then left to right as vectors. The sum is over some fixed complete set of irreducible representations.

It is worth noting that in the case of an abelian group (like the hypercube \( \mathbb{Z}_n^2 \)) the degree of every representation is 1, which significantly simplifies the analysis. We do not enjoy such simplification here.

3 The symmetric group and Young tableaux

The symmetric group on \( n \) elements, which we denote \( S_n \), is the group of permutations \( [n] \to [n] \), where composition is the group operation. We use Greek letters \( \sigma \) and \( \tau \) for elements of \( S_n \).

We will be very interested in the irreducible representations of \( S_n \). We define a partition \( \lambda \) of \( n \) to be a nonincreasing sequence of integers \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \), where \( \sum \lambda_i = n \) and each \( \lambda_i > 0 \). We write \( \lambda \vdash n \) to mean that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) is a partition of \( n \). The following is well-known:

**Theorem 3.1.** The irreducible representations of \( S_n \) are indexed by partitions of \( n \).

It is common to use Ferrer’s diagrams to visualize a partition of \( n \). The Ferrer’s diagrams represent each component of each partition as the number of squares in the corresponding row. A standard Young tableau is a Ferrer’s diagram with the numbers 1, 2, \ldots, \( n \) each occurring in the one cell, such that the numbers in the cells are increasing downwards and to the right. (We refer the reader to [Ful97] for a thorough treatment of Young tableaux.)
Theorem 3.2. The degree of any irreducible representation is equal to the degree of the Ferrer’s diagram of the corresponding partition. That is, \( d_\lambda = d_{\rho_\lambda} \).

We fix ourselves to a specific set of representations known as Young’s Orthogonal Representation. For details on how these representations are constructed, see Appendix B. A consequence of this choice is that for every \( \lambda \vdash n \), the representation \( \rho_\lambda : S_n \to \mathbb{R}^{d_\lambda \times d_\lambda} \) is completely determined by \( \lambda \). It will be convenient to almost exclusively write \( d_\lambda \) and \( \tilde{f}_\lambda \) in place of \( d_\rho \) and \( f_\rho \), since the partition corresponding to the representation is often an important part of the analysis.

Remark 3.3. It follows that we can index the rows and columns of the matrix \( \tilde{f}_\lambda \) by the standard Young tableaux \( T_1, T_2, \ldots, T_{d_\lambda} \) of shape \( \lambda \). The specific ordering we use will often be irrelevant.

We will write \( T \sim \lambda \) to mean that \( T \) is a standard Young tableau of shape \( \lambda \). For \( T \sim \lambda \) with \( \lambda \vdash n \) and \( i \in [n] \), we define \( \text{row}_T(i) \) (\( \text{col}_T(i) \)) to be the row (column) of \( T \) that is filled with cell \( i \). We define the content \( c_T(i) = \text{col}_T(i) - \text{row}_T(i) \); this quantity will arise frequently.

We will make much use of the Shift Theorem for these functions (see the book by Diaconis [Dia88] for details):

Theorem 3.4 (Shift Theorem). Given \( f : S_n \to \mathbb{R} \), let \( f' : S_n \to \mathbb{R} \) be the function defined by \( f'(\sigma) = f(\pi_1 \sigma \pi_2) \) for fixed \( \pi_1, \pi_2 \in S_n \). Then for every \( \lambda \vdash n \), we have \( f'_\rho_\lambda = \rho_\lambda(\pi_1) \hat{f}_\rho_\lambda \rho_\lambda(\pi_2) \).

4 Influence in the symmetric group

We can now give a Fourier theoretic interpretation for the influence. Since we will make heavy use of Fourier analysis, we will change the codomain of Boolean functions from \( \{0, 1\} \) to \( \{-1, 1\} \). Further, we extend our definition to real-valued functions.

Definition 4.1. For a function \( f : S_n \to \mathbb{R} \), we define

\[
\text{Inf}_{ij}(f) = \frac{1}{4} \mathbb{E}[f(\sigma) - f(\sigma_{ij})]^2.
\]

If the codomain of \( f \) is \( \{-1, 1\} \), then the quantity inside the expectation is 0 if \( f(\sigma) = f(\sigma_{ij}) \) and 4 if \( f(\sigma) \neq f(\sigma_{ij}) \). It follows that in this case, \( \text{Inf}_{ij}(f) = \text{Pr}_\sigma[f(\sigma) \neq f(\sigma_{ij})] \), which is the definition in the introduction.

Simple algebra shows that \( \text{Inf}_{ij}(f) = \frac{1}{2} \mathbb{E}_\sigma[f(\sigma)^2] - \frac{1}{2} \mathbb{E}_\sigma[f(\sigma)[f(\sigma_{ij})]] \). Using Plancherel’s Theorem, which for our setting states
\[
E[\sigma(f(g(\sigma))] = \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{g}_\lambda \right)
\]

we apply the Shift Theorem to obtain

\[
E[\sigma(f(\sigma(g(\sigma))) = \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \rho_\lambda((ij)) \right),
\]

and we get the following interpretation of influence

\[
\text{Inf}_{ij}(f) = \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \rho_\lambda((ij)) \right) - \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \rho_\lambda((ij)) \right) = \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \left( I_{d_\lambda \times d_\lambda} - \rho_\lambda((ij)) \right) \right).
\]

where \( I_{d_\lambda \times d_\lambda} \) denotes the identity matrix of order \( d_\lambda \).

One of the hurdles in immediately proving a junta theorem in this domain is the influence of the individual coordinates. It is potentially unwieldy to analyze the effects of subsets of transpositions. Even the influence of a single adjacent transposition might provide complicated results in terms of the Fourier spectrum. In Young’s Orthogonal Representation, even the simplest matrices \( \rho((i, i + 1)) \) are only almost diagonal (each can be written as a block diagonal matrix with blocks of order 1 and 2; see Appendix B for details), and the matrices \( \rho((ij)) \) in general can be quite unwieldy. However, we show that when we have a structured sum of transpositions, the resulting matrix \( \sum \rho((ij)) \) is diagonal.

Since each transposition involves two coordinates, we partition the transpositions into \( n - 1 \) sets, one for each \( j \in \{2, 3, \ldots, n\} \). The set corresponding to \( j \) contains \((ij)\) for \( i < j \). The following proposition is well-known; for example, a proof using Jucys-Murphy elements appears in [VO05]. We give a simple but tedious proof for completeness in Appendix B.

**Proposition 4.2.** Let \( A_\rho(k) = \sum_{j=1}^{k-1} \rho((jk)) \). Then \( A_\rho(k) \) is a diagonal matrix, and the \((T, T)\) entry is \( c_T(k) \).

We note that in particular, \( \sum_{i=1}^{n} A_\rho(i) = \sum_{1 \leq i < j \leq n} \rho_\lambda((ij)) \) is a diagonal matrix whose nonzero entries are \( \sum_{i=1}^{n} c_T(i) \), where \( T \sim \lambda \). When \( \lambda \) is fixed, the sum of contents in any \( T \sim \lambda \) is constant, so \( \sum_{i=1}^{n} A_\rho(i) = (\sum_{i=1}^{n} c_T(i)) I_{d_\lambda \times d_\lambda} \), where \( I_{d_\lambda \times d_\lambda} \) is an identity matrix and \( T \sim \lambda \). It follows that

\[
\text{Inf}(f) = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}(f) = \frac{1}{2} \sum_{1 \leq i < j \leq n} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \left( I_{d_\lambda \times d_\lambda} - \rho_\lambda((ij)) \right) \right)
\]

\[
= \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \sum_{1 \leq i < j \leq n} \left( I_{d_\lambda \times d_\lambda} - \rho_\lambda((ij)) \right) \right) = \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \left( n \choose 2 \right) I_{d_\lambda \times d_\lambda} - \sum_{i=1}^{n} A_\rho(i) \right)
\]

\[
= \frac{1}{2} \sum_{\lambda \vdash n} d_\lambda \left( n \choose 2 \right) - \sum_{i=1}^{n} c_T(i) \right) \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \right) = \sum_{\lambda \vdash n} d_\lambda \frac{1}{2} \left( n \choose 2 \right) - \sum_{i=1}^{n} c_T(i) \right) \| \tilde{f}_\lambda \|^2_F.
\]

Here we have used \( \| \tilde{f}_\lambda \|^2_F = \text{tr} \left( \tilde{f}_\lambda \tilde{f}_\lambda \right) \); the matrix norm here is the Frobenius norm.
We will be interested in the partial sums \( \sum_{i=1}^{m} A_{\rho_i}(i) \) for \( m \leq n \). In this case, the matrix \( \sum_{i=1}^{m} A_{\rho_i}(i) \) is a diagonal matrix whose \((T,T)\) entry for \( T \sim \lambda \) is \( \sum_{i=1}^{m} c_T(i) \). With a similar analysis as above, it follows that

\[
\sum_{1 \leq i < j \leq m} \text{Inf}_{ij}(f) = \frac{1}{2} \sum_{\lambda \vdash n} d_{\lambda} \text{tr} \left( f^T \hat{T}_\lambda \left( \begin{pmatrix} m \\ 2 \end{pmatrix} I_{d_\lambda \times d_\lambda} - \text{diag} \left( \sum_{i=1}^{m} c_T(i) \right) \right) \right)
\]

Further,

\[
\sum_{1 \leq i < j \leq m} \text{Inf}_{ij}(f) = \sum_{\lambda \vdash n} d_{\lambda} \sum_{T \sim \lambda} \left( \frac{1}{2} \begin{pmatrix} m \\ 2 \end{pmatrix} - \frac{1}{2} \sum_{i=1}^{m} c_T(i) \right) \| (\hat{f}_\lambda)_T \|^2
\]

where \( \text{diag}(\sum_{i=1}^{m} c_T(i)) \) denotes a diagonal matrix whose entry in row \( T \) and column \( T \) is \( \sum_{i=1}^{m} c_T(i) \). The ordering on these tableau is the same ordering used in constructing the representation matrices. The vector \( (\hat{f}_\lambda)_T \) is the column of \( \hat{f}_\lambda \) indexed by \( T \), the norm is the standard Euclidean norm. We will frequently refer to the quantity \( \frac{1}{2} \begin{pmatrix} m \\ 2 \end{pmatrix} - \frac{1}{2} \sum_{i=1}^{m} c_T(i) \), so we will make the following definition.

**Definition 4.3.** For a standard Young tableau \( T \sim \lambda \) where \( \lambda \vdash n \) and an integer \( 2 \leq m \leq n \), we define

\[
\beta_T(m) = \frac{1}{2} \begin{pmatrix} m \\ 2 \end{pmatrix} - \frac{1}{2} \sum_{i=1}^{m} c_T(i)
\]

Further, \( \beta_T(n) \) for \( T \sim \lambda \) only depends on \( \lambda \), so define

\[
\beta_\lambda = \beta_T(n) = \frac{1}{2} \begin{pmatrix} n \\ 2 \end{pmatrix} - \frac{1}{2} \sum_{i=1}^{n} c_T(i)
\]

where \( T \) is an arbitrary standard Young tableau of shape \( \lambda \). We also define

\[
\overline{\beta}_\lambda = \beta_\lambda / \begin{pmatrix} n \\ 2 \end{pmatrix}
\]

Thus we can rewrite the previous influence expression in the following way:

\[
\sum_{1 \leq i < j \leq m} \text{Inf}_{ij}(f) = \sum_{\lambda \vdash n} d_{\lambda} \sum_{T \sim \lambda} \beta_T(m) \| (\hat{f}_\lambda)_T \|^2.
\]

yielding the special cases

\[
\text{Inf}(f) = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}(f) = \sum_{\lambda \vdash n} d_{\lambda} \sum_{T \sim \lambda} \beta_T \| (\hat{f}_\lambda)_T \|^2 = \sum_{\lambda \vdash n} d_{\lambda} \beta_\lambda \| \hat{f}_\lambda \|_{F}^2
\]

and

\[
\left( \begin{pmatrix} n \\ 2 \end{pmatrix} \right)^{-1} \text{Inf}(f) = \sum_{\lambda \vdash n} d_{\lambda} \overline{\beta}_\lambda \| \hat{f}_\lambda \|_{F}^2.
\]

The values of \( \beta_\lambda \) and \( \overline{\beta}_\lambda \) have been well-studied in slightly different contexts. For two partitions \( \lambda \vdash n \) and \( \mu \vdash n \), we write \( \lambda \succeq \mu \) to mean that \( \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \) for every integer \( k \), where we append each partition with infinitely many zeros.

**Theorem 4.4** (Diaconis [Dia88]). Suppose \( \lambda \succeq \mu \). Then \( \beta_\lambda \leq \beta_\mu \) and \( \overline{\beta}_\lambda \leq \overline{\beta}_\mu \).
Informally, this result says that representations according to “wide” tableaux contribute little to the influence, while representations according to “tall” tableaux contribute much to the influence. This result can be derived from the work we have already done without too much further effort, but we will not do so here. We will use two corollaries of the above theorem.

**Corollary 4.5.** Let \( \lambda \vdash n \) be such that \( \lambda_1 < n - d \). Then \( \beta_{\lambda} \geq \beta_{(n-d-1,d+1)} \geq \beta_{(n-d,d)} \). By direct calculation, we have

\[
\beta_{(n-d,d)} = \frac{1}{2} \binom{n}{2} - \frac{1}{2} \sum_{i=1}^{n} c_T(i) = \frac{1}{2} d(n-d+1).
\]

**Corollary 4.6.** Let \( \lambda \vdash n \) be such that \( \lambda_1 \geq n - d \). Then \( \overline{\beta}_{\lambda} \leq \overline{\beta}_{(n-d,1^d)} \). By direct calculation, we have

\[
\overline{\beta}_{(n-d,1^d)} = \frac{1}{2} - \frac{1}{2} \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n} c_T(i) = \frac{d}{n-1}.
\]

## 5 Proof of the key result

**Definition 5.1.** We say that \( f : S_n \rightarrow \{-1, 1\} \) is a \( k \)-junta if there exists a set \( S \) of \( k \) coordinates such that

\[
\sum_{i,j \in [n] \setminus S} \Inf_{ij}(f) = 0.
\]

Since \( \Inf_{ij}(f) \geq 0 \), if \( \sum_{i,j \in [n] \setminus S} \Inf_{ij}(f) = 0 \) then \( \Inf_{ij}(f) = 0 \) for all \( i, j \in [n] \setminus S \). If \( \Inf_{ij}(f) > 0 \), then \( f \) depends on at least one of coordinates \( i \) and \( j \). If \( S = \{n-k+1, n-k+2, \ldots, n\} \), then \( f \) only depends on the last \( k \) coordinates if and only if \( \sum_{1 \leq i < j \leq n-k} \Inf_{ij}(f) = 0 \). If \( f \) is a \( k \)-junta, then \( f(\sigma) \) is determined by the values of \( \sigma(s) \) for \( s \in S \). Without loss of generality, we will often assume \( S = \{n-k+1, n-k+2, \ldots n\} \), and say that \( f \) depends on the last \( k \) coordinates. We prove a claim relating Fourier spectrum to the property of being a \( k \)-junta in this way.

**Claim 5.2.** Suppose \( f : S_n \rightarrow \mathbb{R} \) is such that \( (\hat{f}_\lambda)_T \neq 0 \) only if the first row of \( T \) contains \( \{1, 2, \ldots, n-k\} \). Then \( f \) is a \( k \)-junta, and \( f \) only depends on the last \( k \) coordinates.

**Proof.** It suffices to show that \( \sum_{1 \leq i < j \leq n-k} \Inf_{ij}(f) = 0 \). Fix a standard Young tableau \( T \) such that the first row of \( T \) contains \( \{1, 2, \ldots, n-k\} \). By our assumption on the first row of \( T \), \( i \) occurs in the first row and \( i \)th column of \( T \), so \( c_T(i) = i - 1 \) for \( i = 1, 2, \ldots, n-k \). Thus \( \sum_{i=1}^{n-k} c_T(i) = \sum_{i=1}^{n-k} (i-1) = \binom{n-k}{2} \), so \( \beta_T(n-k) = 0 \).

Now for all standard Young tableau \( T \), we have that \( \beta_T(n-k) ||(\hat{f}_\lambda)_T||^2 = 0 \), and it follows from Equation 1 that \( \sum_{1 \leq i < j \leq n-k} \Inf_{ij}(f) = 0. \)

The following proposition says that for a bounded influence function, there is a small set of coordinates that intersects each of the high influence transpositions. This small set will be the coordinates of our junta; equivalently, this set will be the coordinates \( S \) we take in Definition 5.1.

**Proposition 5.3.** Let \( f : S_n \rightarrow \{-1, 1\} \) be a function with \( \Inf(f) \leq \ell n \). For every \( \kappa > 0 \), there exists a set \( S \) of cardinality at most \( 12\ell/\kappa \) with the property that \( \Inf_{ij}(f) \geq \kappa \) implies \( i \in S \) or \( j \in S \).

**Proof.** If \( \ell/\kappa \geq n/12 \), the claim is trivial; take \( S \) to be all \( n \leq 12\ell/\kappa \) coordinates. To handle the case that \( \ell/\kappa < n/12 \), we will first prove a lemma.
Lemma 5.4. For every $f : S_n \to \{-1,1\}$ and every distinct $i,j,k$ such that $1 \leq i,j,k \leq n$ we have $\text{Inf}_{ij}(g) \leq \frac{3}{2}(\text{Inf}_{ik}(g) + \text{Inf}_{jk}(g))$. 

Proof. Note that $(ij) = (ik)(jk)(ik)$ for $k \notin \{i,j\}$, so we have 

$$\Pr[f(\sigma) \neq f(\sigma(ij))] = \Pr[f(\sigma) \neq f(\sigma(ik)(jk)(ik))] \leq \Pr[f(\sigma) \neq f(\sigma(ik))] + \Pr[f(\sigma(ik)) \neq f(\sigma(ik)(jk))] + \Pr[f(\sigma(ik)(jk)) \neq f(\sigma(ik))].$$

Using the fact that $\sigma(ik)$ and $\sigma(ik)(jk)$ are uniformly distributed because $\sigma$ is. Thus the above inequality can be written

$$\text{Inf}_{ij}(f) \leq 2\text{Inf}_{ik}(f) + \text{Inf}_{jk}(f).$$

This argument is symmetric in $i$ and $j$, so we also have $\text{Inf}_{ij}(f) \leq \text{Inf}_{ik}(f) + 2\text{Inf}_{jk}(f)$. Averaging these inequalities gives the lemma. 

We will prove the proposition by contradiction. Suppose no such set $S$ exists. Then we can find $3\ell/\kappa$ pairwise disjoint transpositions (without loss of generality, we can assume the transpositions are adjacent) $(i, i+1)$ such that $\text{Inf}_{i,i+1}(f) \geq \kappa$; these transpositions cover the elements $\{1, 2, \ldots, 6\ell/\kappa\}$. By Lemma 5.4, it follows that $3\kappa \leq 2\text{Inf}_{i,i+1}(g) \leq \text{Inf}_{i,j}(g) + \text{Inf}_{i+1,j}(g)$ for every $1 \leq i \leq 6\ell/\kappa$ with $i$ odd and $6\ell/\kappa < j \leq n$. We have

$$\text{Inf}(g) \geq \sum_{i,j: i \leq 6\ell/\kappa, j > 6\ell/\kappa, i \text{ odd}} \text{Inf}_{ij}(f) \geq (n - 6\ell/\kappa)(3\ell/\kappa)(\frac{2}{3}\kappa) = 2\ell(n - 6\ell/\kappa) > \ell n,$$

a contradiction to our assumption that $\text{Inf}(g) \leq \ell n$. Thus there does not exist a set of $3\ell/\kappa$ pairwise disjoint transpositions with $\text{Inf}_{ij}(g) \geq \kappa$, so in this case there exists a set $S$ of $6\ell/\kappa$ many coordinates such that every transposition with $\text{Inf}_{ij}(g) \geq \kappa$ has at least one index in $S$. 

Analogous to the proof of Friedgut’s junta theorem presented in [O’D07], we want to show that the individual influences of a “smoothed” version of $f$ are significantly smaller than the influences of $f$. To this end, we define the operator $H_t$ such that the function $H_t f(\sigma) = \mathbb{E}_{\tau}[f(\sigma \tau)]$, where $\tau$ is the composition of $m \sim \text{Poisson}(t)$ many random transpositions.

Lemma 5.5. Writing $f = \sum \lambda \text{tr}(f_\lambda \rho_\lambda)$, we have $H_t f = \sum \lambda \exp(-2t \beta_\lambda) \text{tr}(f_\lambda \rho_\lambda).$

Proof. By linearity, it suffices to prove that $H_t \rho_{pq}(\sigma) = \exp(-2t \beta_\lambda) \rho_{pq}(\sigma)$ for every representation $\rho$ and every $1 \leq p, q \leq d_\rho$. Applying the Shift Theorem, we get

$$H_t \rho_{pq}(\sigma) = \mathbb{E}_\tau[\rho_{pq}(\sigma \tau)] = \mathbb{E}_\tau[\rho_{pq}(\sigma) \rho(\tau)] = f(\sigma) \mathbb{E}_\tau[\rho_{pq}(\tau)] = f(\sigma) \mathbb{E}_\tau[\mathbb{E}_m[\rho_{pq}((ij))]^m],$$

where $m \sim \text{Poisson}(t)$ and $(ij)$ is a random transposition. By Proposition 4.2, the inner expectation is

$$\sum_{k=1}^n A_\rho(k) = \sum_{k=1}^n \text{e}^T(k) = 1 - 2\beta_\lambda,$$

where $T \sim \lambda$. Thus, $H_t \rho_{pq}(\sigma) = \mathbb{E}_m[(1 - 2\beta_\lambda)^m]$. Using the moment generating function of Poisson($t$) we get $H_t \rho_{pq}(\sigma) = \exp(t((1 - 2\beta_\lambda) - 1)) f(\sigma) = e^{-2t \beta_\lambda} \rho_{pq}(\sigma)$. 

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It follows that
\[
\sum_{1 \leq i < j \leq m} \text{Inf}_{ij}(H_i f) = \sum_{\lambda \vdash n} d_\lambda \exp(-4t\beta_\lambda) \sum_{T \sim \lambda} \beta_T(m) \| (\hat{f}_\lambda) T \|^2
\]

Equipped with this expression, we proceed to the key result for the proof of the main theorem.

**Theorem 5.6.** Assume that \( f : S_n \to \{-1, 1\} \) satisfies (i) \( \text{Inf}(f) \leq \ell n \) and (ii) \( \text{Inf}_{ij}(H_i f) \leq 2(\text{Inf}_{ij}(f))^{3/2} \) where \( t = (\ln 3)/(2\alpha) \), for all \( 1 \leq i < j \leq n \). Set \( d = 3\ell /\epsilon \) and \( K = 3^{5/(n\alpha)}e^4 \). Then there exists a function \( h : S_n \to \{-1, 1\} \) such that \( \Pr_{\sigma}[f(\sigma) \neq h(\sigma)] \leq \epsilon \) and \( h \) only depends on \( 60\ell K^d \) coordinates.

The proof involves the following steps:

1. **Proof.** Set \( R = 12\ell K^d \). If \( R > n/5 \), then \( n < 5R = 60\ell K^d \) and the theorem is trivial. From this point forward, we assume \( R \leq n/5 \).

2. Let \( T' \) be the set of standard Young tableaux of shape \( \lambda \) such that \( \lambda_1 < n - d \), and let \( T'' \) be the set of standard Young tableaux of shape \( \lambda \) such that \( \lambda_1 \geq n - d \) and at least one element of \( \{1, 2, \ldots, n-R\} \) occurs in the second row. We will show that \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T'} \| (\hat{f}_\lambda) T \|^2 \leq \epsilon/2 \) and \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T''} d_\lambda \| (\hat{f}_\lambda) T \|^2 \leq \epsilon/2 \), where we have suppressed the condition that \( T \sim \lambda \) in both sums. We show later that this is sufficient to construct a k-junta approximating \( f \).

3. The first of these bounds is quite straightforward. Since \( R \leq n/5 \), \( d \) is certainly less than \( n/5 \). It follows that \( \frac{1}{2} (n - d + 1) > 2n/3 \). Using Corollary 4.5, for \( T \in T' \), we have \( \beta_\lambda \geq \frac{1}{2} d(n - d + 1) \) for \( T \sim \lambda \). Thus, if \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T'} \| (\hat{f}_\lambda) T \|^2 \) is not greater than \( \epsilon/2 \), then by Equation 2 we have

\[
\text{Inf}(f) > (\epsilon/2)(3\ell /\epsilon)(n - 3\ell /\epsilon + 1)/2 \geq (3\ell /2)(2n/3) = \ell n,
\]

which is a contradiction to our assumption. Thus \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T'} \| (\hat{f}_\lambda) T \|^2 \leq \epsilon/2 \).

4. The more involved case is showing \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T''} \| (\hat{f}_\lambda) T \|^2 \leq \epsilon/2 \). First, a lemma.

**Lemma 5.7.** For a tableau \( T \in T'' \) of shape \( \lambda \) with \( n - R \leq \lambda_1 < n \), we have \( \beta_T(n - R) \geq n/3 \) assuming \( R \leq n/5 \).

**Proof.** The cells numbered 1, 2, . . ., \( n - R \) form a tableau on their own, which we will call \( T_1 \). The tableau \( T_1 \) that minimizes \( \beta_T(n - R) = \beta_{T_1}(n - R) \) (that is a “subtableau” of \( T \in T'' \)) has shape \( (n - R - 1, 1) \) by Theorem 4.4. For such a tableau \( T_1 \), it is straightforward to verify that \( \beta_T(n - R) = \frac{1}{2} \left( \frac{n - R}{2} \right) - \frac{1}{2} \sum_{p=1}^{n-R} c_T(p) = (n - R)/2 \). Assuming \( R \leq n/5 \), we have \( (n - R)/2 \geq 2n/5 \geq n/3 \).

The lemma above implies that \( \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T''} \| (\hat{f}_\lambda) T \|^2 \leq \frac{3}{n} \sum_{\lambda \vdash n} d_\lambda \sum_{T \in T''} \beta_T(n - R) \| (\hat{f}_\lambda) T \|^2 \).

Using Corollary 4.6, we have that \( \beta_\lambda \leq \beta_{(n - d, 1)^d} = d/(n - 1) \) when \( \lambda_1 \geq n - d \). This is the case for every \( \lambda \) such that \( T \sim \lambda \) for some \( T \in T'' \), so the above expression is at least
\[
\frac{3}{n} \exp\left(4t\beta_{(n-d,1)}\right) \sum_{\lambda \geq n} d_\lambda \exp\left(-4t\beta_\lambda\right) \sum_{T \in \mathcal{T}', T \sim \lambda} \beta_T(n - R)\|\hat{f}_\lambda_T\|^2 \leq \\
\frac{3}{n} \exp\left(4t \frac{d}{n-1}\right) \sum_{\lambda \geq n} d_\lambda \exp\left(-4t\beta_\lambda\right) \sum_{T \sim \lambda} \beta_T(n - R)\|\hat{f}_\lambda_T\|^2 = \\
\frac{3}{n} \exp\left(4t \frac{d}{n-1}\right) \sum_{1 \leq i < j \leq n - R} \text{Inf}_{ij}(H_if)
\]

By assumption, we have

\[
\text{Inf}_{ij}(H_if) \leq 2(\text{Inf}_{ij}(f))^{3/2} \leq 2\text{Inf}_{ij}(f)(\max_{1 \leq i < j \leq n - R} \text{Inf}_{ij}(f)^{1/2})
\]

when \(1 \leq i < j \leq n - R\) and \(t = \frac{\ln 3}{2} \frac{1}{\alpha}\). Applying Proposition 5.3 with \(\kappa = K^{-d}\) (recall \(R = 12\ell K^d\)), we have that \(\text{Inf}_{ij}(f) \leq K^{-d}\) for \(1 \leq i < j \leq n - R\). Continuing, we get

\[
(*) \leq \frac{6}{n} \exp\left(4t \frac{d}{n-1}\right) \left(\max_{1 \leq i < j \leq n - R} \text{Inf}_{ij}(f)^{1/2}\right) \sum_{1 \leq i < j \leq n - R} \text{Inf}_{ij}(f) \\
\leq \frac{6}{n} \exp\left(4t \frac{d}{n-1}\right) \text{Inf}(f) K^{-d/2} \\
\leq 6 \exp\left(4t \frac{d}{n-1}\right) \ell K^{-d/2} \\
= 6 \cdot 3^{5d/(2n\alpha)} \ell K^{-d/2} \\
\leq 6 \ell \exp(-2d) = 6 \ell \exp(-6\ell/\epsilon)
\]

Using the fact that \(x \leq \exp(x/2)\) and thus \(\exp(-x) \leq 1/(2x)\) when \(x \geq 0\), we have \(6\ell \exp(-6\ell/\epsilon) \leq 6\ell\epsilon/(12\ell) = \epsilon/2\).

Let \(h : S_n \to \mathbb{R}\) be the function such that

\[
(\hat{h}_\lambda)_T = \begin{cases} 
(\hat{f}_\lambda)_T & \text{if } T \notin \mathcal{T}' \cup \mathcal{T}'' \\
0 & \text{if } T \in \mathcal{T}' \cup \mathcal{T}''
\end{cases}
\]

By construction, \((\hat{h}_\lambda)_T \neq 0\) only if the first row of \(T\) contains \(\{1, 2, \ldots, n - R\}\), so it follows from Claim 5.2 that \(h\) (and thus \(\text{sgn}(h)\)) is an \(R\)-junta and depends only on the last \(R\) coordinates.

We recall Plancherel, which for \(f = g\) reduces to

\[
\mathbb{E}[f(\sigma)^2] = \sum_{\lambda \geq n} d_\lambda \sum_{T \sim \lambda} \|\hat{f}_\lambda_T\|^2.
\]

It follows that \(\mathbb{E}_\sigma[(f(\sigma) - h(\sigma))^2] = \sum_{\lambda \geq n} d_\lambda \sum_{T \in \mathcal{T}' \cup \mathcal{T}''} \|\hat{f}_\lambda_T\|^2 \leq \epsilon\). The theorem follows from noting \(\mathbb{P}_\sigma[f(\sigma) \neq \text{sgn}(h(\sigma))] \leq \mathbb{E}_\sigma[(f(\sigma) - h(\sigma))^2],\) since \(f(\sigma) \neq \text{sgn}(h(\sigma))\) implies \(|f(\sigma) - h(\sigma)| \geq 1\) and \((f(\sigma) - h(\sigma))^2 \geq 1\). Thus \(\mathbb{P}[f(\sigma) \neq \text{sgn}(h(\sigma))] \leq \epsilon\) and \(\text{sgn}(h)\) depends only on the last \(R\) coordinates, completing the proof. \(\Box\)
6 Application to slices of the hypercube

We will apply Theorem 5.6 to prove the main theorem. In this section, we will be concerned with functions $g : \binom{[n]}{k} \to \{-1, 1\}$. Our method is to lift such functions to functions over the symmetric group in a way that preserves the added structure.

To this end, we define the map $P_k : S_n \to \binom{[n]}{k}$ such that $(P_k(\sigma))_i = 1$ if $\sigma(i) \geq k$ and $(P_k(\sigma))_i = 0$ otherwise. Note that $P_k$ is a $k!(n-k)!$-to-one map, so the distribution of $P_k(\sigma)$ for a uniform random permutation $\sigma$ is the uniform distribution over $\binom{[n]}{k}$. It is also easy to verify that $P_k(\sigma(ij)) = P_k(\sigma)(ij)$ for all $\sigma \in S_n$ and all $i$ and $j$. For any $x \in \binom{[n]}{k}$, and $\sigma \in S_n$, we define $x\sigma$ such that $(x\sigma)_i = x_{\sigma(i)}$.

Slightly abusing notation, we define

$$\text{Inf}_{ij}(g) = \frac{1}{4} \mathbb{E}[|g(x) - g(x(ij))|^2]$$

and

$$\text{Inf}(g) = \sum_{1 \leq i < j \leq n} \text{Inf}_{ij}(g)$$

where $x(ij)$ denotes the string $x$ with the $i$th and $j$th bits transposed. (It is possible that transposing these bits has no effect.) Also, we define the operator $H_tg$ such that $g(x) = \mathbb{E}_{\tau}[g(x\tau)]$, where $\tau$ is the composition of $m \sim \text{Poisson}(t)$ many random transpositions.

For every $g : \binom{[n]}{k} \to \mathbb{R}$, define the function $f^g : S_n \to \mathbb{R}$ such that $f^g(\sigma) = g(P_k\sigma)$. The following proposition gives the motivation for our abuse of notation.

**Proposition 6.1.** We have (i) $\text{Inf}_{ij}(f^g) = \text{Inf}_{ij}(g)$, (ii) $f^{H_tg} = H_tf^g$, and (iii) $\text{Inf}_{ij}(H_tf^g) = \text{Inf}_{ij}(H_tf^g)$ for all $g : \binom{[n]}{k} \to \mathbb{R}$.

**Proof.** The proof of (i) is straightforward:

$$\text{Inf}_{ij}(f^g) = \frac{1}{4} \mathbb{E}_\sigma[|f^g(\sigma) - f^g(\sigma(ij))|^2]$$

$$= \frac{1}{4} \mathbb{E}_\sigma[|g(P_k(\sigma)) - g(P_k(\sigma(ij)))|^2]$$

$$= \frac{1}{4} \mathbb{E}_\sigma[|g(P_k(\sigma)) - g(P_k(\sigma)(ij))|^2]$$

$$= \frac{1}{4} \mathbb{E}_\sigma[|g(x) - g(x(ij))|^2] = \text{Inf}_{ij}(g)$$

For (ii), let $\tau$ be a sequence of $m$ many random transpositions, where $m \sim \text{Poisson}(t)$. Then for all $\sigma$, we have $H_tf^g(\sigma) = \mathbb{E}_{\tau}[f^g(\sigma\tau)] = \mathbb{E}_{\tau}[g(P_k(\sigma\tau))] = \mathbb{E}_{\tau}[g(P_k(\sigma))\tau] = (H_tg)(P_k(\sigma)) = f^{H_tg}(\sigma)$, where we applied the fact that $P_k(\sigma(ij)) = P_k(\sigma)(ij)$ $m$ times.

Statement (iii) follows from applying (i) then (ii): $\text{Inf}_{ij}(H_tf^g) = \text{Inf}_{ij}(f^{H_tg}) = \text{Inf}_{ij}(H_tf^g)$.

**Lemma 6.2.** The following are equivalent:

(i) $f : S_n \to \mathbb{R}$ satisfies $f_\lambda = \rho_\lambda((ij))f_\lambda$ for all $\lambda \vdash n$ and for all $1 \leq i < j \leq k$ and $k+1 \leq i < j \leq n$.

(ii) There exists $g : \binom{[n]}{k} \to \mathbb{R}$ such that $f = f^g$.

**Proof.** Note that $P_k(\sigma)$ is completely determined by the preimages of $[k]$ under $\sigma$, which we will denote $\sigma^{-1}([k])$. Since $P_k(\sigma)$ is completely determined by $\sigma^{-1}([k])$, $P_k((ij)\sigma) = P_k(\sigma)$ if $1 \leq i < j \leq k$. Since $P_k(\sigma)$ is also completely determined by $\sigma^{-1}([n] \setminus [k])$, $P_k((ij)\sigma) = P_k(\sigma)$ if $k+1 \leq i < j \leq n$. 


To show (ii) implies (i), we have \(f^g(\sigma) = f^g((ij)\sigma)\) if \(i\) and \(j\) are both at most \(k\) or both at least \(k + 1\). It follows from the Shift Theorem that \(\hat{f}_\lambda^g = \rho_\lambda((ij))\hat{f}_\lambda^g\) for all \(\lambda \vdash n\) and all \((ij)\) where \(1 \leq i < j \leq k\) or \(k + 1 \leq i < j \leq n\).

We can reverse this argument to show that (i) implies (ii). Suppose \(f : S_n \to \mathbb{R}\) satisfies \(\hat{f}_\lambda = \rho_\lambda((ij))\hat{f}_\lambda\) for all \(1 \leq i < j \leq k\) and \(k + 1 \leq i < j \leq n\) and all \(\lambda \vdash n\). Then by the Shift Theorem \(f(\sigma) = f((ij)\sigma)\) for all such transpositions \((ij)\). We can conclude that \(f\) is constant on \(P_k(\sigma)\) for all \(\sigma\), and there exists a function \(g : \binom{n}{k} \to \{-1, 1\}\) such that \(f = f^g\).

The proof in Theorem 2.6 of [OW09] shows that for a function \(g' : \binom{n}{k} \to \{0, 1\}\),

\[
2\text{Inf}_{ij}(H_tg') \leq (4\text{Inf}_{ij}(g'))^{3/2}
\]

for all \(1 \leq i < j \leq n\), where \(t = (\ln 3)/(2\alpha_k)\). To apply to our case, we want the range to be \(\{-1, 1\}\); to apply to a function \(g : \binom{n}{k} \to \{-1, 1\}\), we define \(g' = \frac{1}{2}(g + 1)\) and apply the above, yielding

\[
\text{Inf}_{ij}(H_tg) = 4\text{Inf}_{ij}(H_t\frac{1}{2}(g' + 1)) \leq 2(4\text{Inf}_{ij}(\frac{1}{2}(g' + 1)))^{3/2} = 2(\text{Inf}_{ij}(g))^{3/2}
\]

Applying this result which in turns relies on a result of Lee and Yau [LY98], we can take the value of \(\alpha_k\) to be \(C/(n \log(1/\nu(k)))\) for some universal constant \(C\), where \(\nu(k) = k(n - k)/\binom{n}{k}\).

**Theorem 6.3** (Theorem 1.3 restated). Assume that \(g : \binom{n}{k} \to \{-1, 1\}\) has \(\text{Inf}(g) \leq \ell n\). For \(\epsilon > 0\), set \(d = 3\ell/\epsilon\) and \(K = 3^{\ell/(n - \alpha_k)}\). Then there is a function \(h : \binom{n}{k} \to \{-1, 1\}\) such that \(\Pr_x[g(x) \neq h(x)] \leq \epsilon\) and \(h\) depends only on \(60\ell K^d\) coordinates.

When \(k/n\) is bounded away from 0 and 1, we can set \(\alpha_k\) such that \(K^d = 2^{O(\text{Inf}(g)/\epsilon)}\).

**Proof.** We now apply Theorem 5.6 to \(f^g\) with \(\alpha = \alpha_k = C/(n \log(1/\nu(k)))\). The function \(f^g\) satisfies the hypotheses of the theorem, since for \(1 \leq i < j \leq n\) and \(t = (\ln 3)/(2\alpha_k)\) we have

\[
\text{Inf}_{ij}(H_tf^g) = \text{Inf}_{ij}(H_tg) \leq 2(\text{Inf}_{ij}(g))^{3/2} = 2(\text{Inf}_{ij}(f^g))^{3/2}
\]

by Proposition 6.1 and [OW09]. Theorem 5.6 promises the existence of a function \(h : S_n \to \mathbb{R}\) such that \(h\) and \(h' := \text{sgn}(h)\) depends on the last \(60\ell K^d\) coordinates, and \(\Pr_x[f^g(\sigma) \neq h'(\sigma)] \leq \epsilon\). The proof of Theorem 5.6 shows that every column of \(\hat{h}_\lambda\) is either the corresponding column of \(\hat{f}_\lambda^g\) or 0. Thus we can apply Lemma 6.2 to see that \(h = f^{g'}\) for some \(g' : \binom{n}{k} \to \mathbb{R}\). Then \(h' = \text{sgn}(h) = \text{sgn}(f^{g'}) = f^{\text{sgn}(g')},\) so

\[
\Pr_x[g(x) \neq \text{sgn}(g')(x)] = \Pr_\sigma[f^g(\sigma) \neq f^{\text{sgn}(g')}(\sigma)] \leq \epsilon,
\]

completing the proof.

## 7 Conclusion

We have proved a stability result for functions over slices of the hypercube. This junta theorem can be stated combinatorially. A tantalizing future direction is the application of this theorem and these techniques to purely combinatorial questions over the Boolean hypercube, slices of the Boolean hypercube, and permutations. Further, it would be of great interest to develop a analogue of Friedgut’s junta theorem covering more general nonproduct settings.
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References


We work on Schreier graphs $\text{Sch}(G, X, U)$, where $G$ is generated by a set $U$ which is assumed to be symmetric and closed under conjugation, and $G$ acts on $X$ in some way; we write $x^g$ for the action of $g \in G$ on $x \in X$. If $X = G$ and the action is the group operation, then we recover the Cayley graph $\text{Cay}(G, U)$.

We define the operators $L_u f(x) = f(x) - f(x^u)$, $L = \frac{1}{|U|} \sum_{u \in U} L_u$. We define $\text{Inf}_u(f) = \frac{1}{4} \mathbb{E}_x [L_u f(x)^2] = \frac{1}{4} \mathbb{E}_x [(f(x) - f(x^u))^2]$, which is consistent with our previous definition, although it differs from the definition in [OW09] by a factor of 2. We define the inner product $\langle f, g \rangle = \mathbb{E}_x [f(x)g(x)]$; it follows that $\text{Inf}_u(f) = \frac{1}{4} \langle L_u f, L_u f \rangle = \frac{1}{2} \langle L_u f, f \rangle$. We write $\| f \|_2^2 = \langle f, f \rangle$.

We can also write $L = I - K$, where $I$ is the identity operator and $K$ is the transition operator for the natural random walk on $\text{Sch}(G, X, U)$.

We further define $H_t = \exp(-tL)$ which acts as a smoothing operator (these operators for $t$ in $\mathbb{R}^{\geq 0}$ form the continuous time Markov semigroup). We see that

$$H_t = \exp(-tL) = \exp(-t) \exp(K) = \sum_{m=0}^{\infty} \frac{e^{-t} t^m}{m!} K^m$$
From this expression, we can see that $H_t f(x) = \mathbb{E}_y[f(y)]$, where $y$ is formed by taking $m \sim \text{Poisson}(t)$ random steps from $x$. For our application here, we always have $G = S_n$ and $U$ the set of transpositions, so we can write this as $H_t f(x) = \mathbb{E}_\tau[f(x^\tau)]$, where $\tau$ is the composition of $m \sim \text{Poisson}(t)$ many random transpositions. This is the definition we use in the rest of the paper.

The generalized KKL Theorem from [OW09] proceeds by upper and lower bounding $\text{Inf}_u(H_t f)$. (This quantity is written $\|L_u H_t f\|_2^2$ in [OW09].) We don’t require the entire KKL Theorem here, we only need the upper bound from the proof of their Theorem 2.6: for $f : X \to \{0, 1\}$ and all $u \in U$, we have

$$2\text{Inf}_u(H_t f) \leq (4\text{Inf}_u(f))^{3/2}$$

where $t = \frac{\ln 3}{2} \cdot \frac{1}{\alpha}$, $\alpha$ denotes the log-Sobolev constant, and we have taken the factor of 2 difference in definitions of influence into account. The log-Sobolev constant $\alpha$ is the largest value such that for all nonconstant $f : X \to \mathbb{R}$, we have

$$\frac{1}{|U|} \text{Inf}(f) \geq \alpha (\mathbb{E}[f(x)^2 \log f(x)^2] - \mathbb{E}[f(x)^2] \log \mathbb{E}[f(x)^2])$$

and a lower bound on the log-Sobolev constant implies some level of hypercontractivity of $H_t$ (see the work of [Gro75] or [DSC96] for details).

Consider the eigenvalue/eigenfunction decomposition of the Laplacian $L$. It is well-known that $L$ has nonnegative real eigenvalues denoted

$$0 = \phi_0 < \phi_1 \leq \phi_2 \leq \cdots \leq \phi_{|X|-1}$$

We write $(\psi_i)_{i=0}^{|X|-1}$ for corresponding eigenfunctions forming an orthonormal basis of $L^2(X)$, with $\psi_0 \equiv 1$. Note that the $\psi_i$’s are also eigenfunctions for the operator $H_t$ with eigenvalues $\exp(-t\phi_i)$. For a given $f \in L^2(X)$ we write $f^i$ for its projection onto the $i$th eigenspace, $f^i = \langle f, \psi_i \rangle \psi_i$. Then

$$\text{Inf}(f) = \left[ \frac{|U|}{2} \sum_{i=0}^{|X|-1} \phi_i \|f^i\|_2^2 \right] \quad \text{and} \quad \text{Inf}(H_t f) = \left[ \frac{|U|}{2} \sum_{i=0}^{|X|-1} \phi_i \exp(-2t\phi_i) \|f^i\|_2^2 \right].$$

It is known (see for example, [Dia88]) that for finite groups, the group representations of $G$ can be taken to be the eigenfunctions above. In the case of the Cayley graph of $S_n$ generated by transpositions, we get that each representation $\rho$ contributes $(d_\rho)^2$ eigenfunctions, all with the same eigenvalue. Letting $\phi_\lambda$ be common eigenvalue, these eigenvalues satisfy $\phi_\lambda = 2\beta_\lambda$, where $\beta_\lambda$ appears in the main narrative.

### B Young’s Orthogonal Representation and a proof of Proposition 4.2

Here, we will demonstrate the properties of a specific set of representations known as Young’s Orthogonal Representation (also known as the Gelfand-Tsetlin basis) that we will use. These properties are well-known, an accessible introduction can be found in [HGG09]. Part of the beauty of Young’s Orthogonal Representation is that the representation matrices are particularly easy to construct for adjacent transpositions. Using these matrices as building blocks, we can construct the representation matrix at any permutation via a procedure similar to bubble sort, recalling that $\rho$ is a representation and that the symmetric group is generated by the adjacent transpositions.
We give the details now. Suppose \( \lambda \vdash n \), and let \( \rho \) be the corresponding representation matrix. We index the rows of columns of \( \rho \) by standard Young tableaux of shape \( \lambda \); the ordering of these tableaux is irrelevant. Given a standard Young tableau \( T \), we define \((i,j) \circ T\) to be the tableau where \( i \) and \( j \) are interchanged. This operation does not always result in a standard tableau.

We recall some definitions concerning standard Young tableaux. We have previously defined the content of \( i \) in \( T \), denoted \( c_T(i) \), to be \( \text{row}_T(i) - \text{col}_T(i) \). These quantities are natural; we define \( \text{row}_T(i) \) (\( \text{col}_T(i) \)) to be the row (column) of the cell that is filled with \( i \). The axial distance from \( i \) to \( j \) is \( d_T(i,j) = (\text{col}_T(i) - \text{col}_T(j)) - (\text{row}_T(i) - \text{row}_T(j)) = c_T(j) - c_T(i) \). Note that this is a signed distance; the axial distance can be negative, and \( d_T(i,j) = -d_T(j,i) \).

We are now ready to describe the representation matrices corresponding to adjacent transpositions. The matrix \( \rho((i,i + 1)) \) consists of all zeros, except in the following cases:

1. The main diagonal entry \((T,T)\) is \( \frac{1}{d_T(i,i + 1)} \).

2. If \((i,i + 1) \circ T\) results in the standard tableau \( T' \), then the \((T,T')\) and \((T',T)\) entries are \( \sqrt{1 - \frac{1}{d_T(i,i + 1)^2}} \).

Note that if \((i,i + 1) \circ T = T'\) then \((i,i + 1) \circ T' = T\), and we get the same condition.

Note that in the former case, the \((T,T)\) entry is equal to \( \pm 1 \) if and only if \((i,i + 1) \circ T\) is not a standard tableau. It follows that \( \rho((i,i + 1)) \) is an orthogonal matrix. If we rearrange the rows and columns of \( \rho((i,i + 1)) \), we get a block diagonal matrix whose blocks are square matrices of order at most 2. In this rearranging, we require that the rows and columns have the same ordering.

One useful aspect of this set of representations is that it behaves very nicely in terms of the subgroup chain \( S_n > S_{n-1} > \cdots > S_2 > S_1 \). Here, we define \( S_k \) as the set of permutations of \([n]\) that fix all elements in \([n] \setminus [k] \); this allows us to treat the symmetric groups as a chain of subgroups. Thus, for most of our analysis, we will assume that the “influential” coordinates as in Friedgut’s junta theorem are the last coordinates. We can clearly assume this by simply relabeling the coordinates.

**Proof of Proposition 4.2.** By induction on \( k \). For \( k = 1 \) the statement is immediate; \( c_T(1) = 0 \) in every standard Young tableau, so the resulting matrix is the zero matrix.

For the inductive case, we appeal to the recursive relationship relating \( A_\rho(k + 1) \) to \( A_\rho(k) \). Thus by the Shift Theorem,

\[
A_\rho(k + 1) = \rho((k,k + 1))A_\rho(k)\rho((k,k + 1)) + \rho((k,k + 1)).
\]

To see that this relation holds, multiplying on the left and right by \((k,k + 1)\) renames coordinate \( k \) to \( k + 1 \) in the \( k - 1 \) transpositions \((1k), (2k), \ldots, (k - 1,k)\). Adding \((k,k + 1)\) finishes the relation.

Assume without loss of generality that the rows and columns of these matrices are permuted so that row \( T \) is adjacent to \((k,k + 1) \circ T\) if \((k,k + 1) \circ T\) is a standard tableau. This implies that \( \rho((k,k + 1)) \) is a block-diagonal matrix, whose blocks are square matrices of order 1 (in the case that \( k \) and \( k + 1 \) are adjacent in \( T \)) or 2. It follows that \( A_\rho(k + 1) \) will be a block diagonal matrix of the same structure.

We work by cases:

**Case 1** \((k,k + 1) \circ T\) is not a standard Young tableau. In this case, the \((T,T)\) entry of \( \rho((k,k + 1)) \) is 1 if \( k \) is to the left of \( k + 1 \) and -1 if \( k \) is above \( k + 1 \). In either case, this entry is \( c_T(k + 1) - c_T(k) \). By induction, the \((T,T)\) entry of \( A_\rho(k + 1) \) will be \( (c_T(k + 1) - c_T(k))^2c_T(k) + c_T(k + 1) = c_T(k + 1) \) (since \( c_T(k + 1) - c_T(k) \) is \( \pm 1 \)), completing the inductive step in this case.
Case 2 $(k, k + 1) \circ T$ is the standard Young tableau $T'$. Letting $d = d_T(k, k + 1)$, the $2 \times 2$ matrix corresponding to the rows and columns of $\rho((k, k + 1))$ corresponding to $T$ and $T'$ is
\[
\begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix},
\]
where we have assumed without loss of generality that $T$ precedes $T'$.

Now the $2 \times 2$ matrix corresponding to the $T$ and $T'$ rows and columns of $A_\rho(k + 1)$ is
\[
\begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} \begin{bmatrix}
ct(k) & 0 \\
0 & c_t(k)
\end{bmatrix} \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} + \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} = \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} \begin{bmatrix}
c_T(k)/d & ct(k)/d \\
c_T(k)/d & -ct(k)/d
\end{bmatrix} + \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} = \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} \begin{bmatrix}
c_T(k)/d^2 + ct(k)/(1 - 1/d^2) & (c_T(k) - ct(k))/\sqrt{1 - 1/d^2}/d \\
(c_T(k) - ct(k))/(1 - 1/d^2) & ct(k)/(1 - 1/d^2)
\end{bmatrix} + \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} = \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} \begin{bmatrix}
(c_T(k) - ct(k))/d^2 + ct(k)/(1 - 1/d^2) & (c_T(k) - ct(k))/\sqrt{1 - 1/d^2}/d \\
(c_T(k) - ct(k))/(1 - 1/d^2) & ct(k)/(1 - 1/d^2)
\end{bmatrix} + \begin{bmatrix}
\frac{1}{d} & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix}
\]

Using $c_t'(k) = c_t(k + 1)$, $c_t(k) = c_t(k + 1)$, and $d = d_T(k, k + 1) = c_t(k + 1) - c_t(k) = c_t'(k) - c_t(k) = -(c_t(k) - c_t'(k))$, the above is equivalent to
\[
\begin{bmatrix}
c_t(k) - 1/d & \sqrt{1 - 1/d^2} \\
-\sqrt{1 - 1/d^2} & c_t(k) + 1/d
\end{bmatrix} + \begin{bmatrix}
1/d & \sqrt{1 - 1/d^2} \\
\sqrt{1 - 1/d^2} & -1/d
\end{bmatrix} = \begin{bmatrix}
c_t'(k) & 0 \\
0 & c_t(k)
\end{bmatrix} = \begin{bmatrix}
c_t(k + 1) & 0 \\
0 & c_t'(k + 1)
\end{bmatrix},
\]
completing the inductive step in this case.

\[\square\]

C Tightness of Theorem 5.6

\textbf{Theorem C.1} (Theorem 1.4). There exists a function $f : S_n \to \{-1, 1\}$ such that (i) $\inf(f) \leq 2n$, and (ii) any function $g : S_n \to \{-1, 1\}$ that only depends on $n/12$ coordinates has $\Pr[f(\sigma) \neq g(\sigma)] \geq 1/4$.

\textbf{Proof}. We use the same function as in Theorem 3 of [OW13], which shows that Talagrand’s extension of KKL cannot be substantially improved for Boolean functions over $S_n$. Specifically, let $f : S_n \to \{0, 1\}$ be the function such that $f(\sigma) = 1$ if $\sigma$ contains a fixed point and $f(\sigma) = 0$ if $\sigma$ is a derangement. It is well-known that $1/3 \leq \Pr[f = 0] \leq 1/2$. It is shown in [OW13] that $\inf(f) \leq 2n$.

Without loss of generality, let $g : S_n \to \{0, 1\}$ be the function over the last $M$ coordinates such that $\Pr[f \neq g]$ is as small as possible. For all $i$, the probability that $i$ is a fixed point is $1/n$, so by the union bound, the probability that none of the last $M$ coordinates contain a fixed point is at least $1 - M/n$. Let $M$ be a partial permutation that only assigns values to the last $M$ coordinates, and let $f|_M$ be the restriction of $f$ to permutations consistent with $M$. Since $g$ only depends on the last $M$ coordinates, $g|_M$ is constant. Thus the minimum error of $g|_M$ occurs when $g|_M \equiv \arg\max_b \Pr[f|_M = b]$ for all $M$. 

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By renaming the coordinates, it is easy to see that for two partial permutations $\mathcal{M}_1$ and $\mathcal{M}_2$, we have $\Pr[f|\mathcal{M}_1 = 0] = \Pr[f|\mathcal{M}_2 = 0]$ if $\mathcal{M}_1$ and $\mathcal{M}_2$ do not contain fixed points. Over a random choice of $\mathcal{M}$, the probability that $\mathcal{M}$ contains no fixed points is at least $1 - M/n$, so either $\Pr[g = 0]$ or $\Pr[g = 1]$ is at least $1 - M/n$. But $\max\{\Pr[f = 0], \Pr[f = 1]\} \leq 2/3$, so

$$\Pr[f \neq g] \geq \max_b \Pr[g = b] - \max_b \Pr[f = b] \geq 1/3 - M/n.$$ 

It follows that if $\Pr[f \neq g] \leq 1/4$, then $M \geq n/12$. \qed