Problems 1

Problem 1 Razulio's Monster Smoothy company sells three kinds of fruit smoothies. Each smoothy consists of some combination of apples, oranges, and bananas. Smoothy 1 has 2 apples, seven oranges, and three bananas. Smoothy 2 has three apples, five oranges, and two bananas. Smoothy 3 has one apple, nine oranges, and four bananas. Each kind of fruit has a certain unit cost.

a) Is there any unit pricing by which the first smoothy would cost \$24, the second would cost \$20, and the third would cost \$15?

b) Suppose that smoothies one, two, and three cost \$24, \$20, and \$28, respectively. Estimate (to the nearest cent) the cost of each piece of fruit.

Problem 2 Suppose the finite Laplace transform

$$F(s) = \int_0^1 e^{-su} f(u)du, \qquad 0 \leq s \leq 1$$

is discretized to produce the $n \times n$ matrix A with $a_{ij} = e^{-ij/n^2}$, for $i, j = 1, \cdots, n$. Find $cond(A)$ for n = 10, 20, 50. For each such N, generate an $n-$vector $\mathbf{b}$ with random components in $[-1, 1]$ and compute the vector $\mathbf{b} = A\mathbf{x}$. Plot the vector $\mathbf{x}$ and the vector $\mathbf{b}$. Explain the results.

Problem 3 Suppose $\mathbf{b} = [1, 0, 2]^T$ and

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Show that the system $A\mathbf{x} = \mathbf{b}$ has no ordinary solution, but that it does have infinitely many least-squares solutions. Find the least-squares solution that is orthogonal to the null space.

Problem 4 Given an arbitrarily small positive number ϵ, construct a matrix A with $\det A = \epsilon$ and $cond(A) = 1 + \epsilon$.

Problem 5 Let A be a real matrix.

a) Show that $\mathcal{N}(A) = \mathcal{N}(A^T A)$.

b) Show that $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$.

Problem 6 Let $P_3[0, 1]$ be the space of polynomials of degree less than or equal to 3 on the interval $[0, 1]$. The polynomials $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$, and $p_4(x) = x^3$ form a basis for $P_3[0, 1]$, but they are not orthogonal with respect to the inner product

$$f \cdot g = \int_0^1 f(x)g(x)dx.$$

Use the Gram-Schmidt orthogonalization process to construct an orthogonal basis for $P_3[0, 1]$. Once you have your basis, use it to find the third-degree polynomial that best approximates $f(x) = e^{-x}$ on the interval $[0, 1]$.

Problem 7 Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector 45 degrees in the clockwise direction. Write down the matrix representation of M with respect to the basis $e_1 = (1, 1)$ and $e_2 = (0, 2)$.

Problem 8 Consider the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ and $\mathbf{b}$ lie in $\mathbb{R}^2$.

a) Suppose $A = \begin{bmatrix} 0.25 & 0.43 \\ 0.43 & 0.75 \end{bmatrix}$ and $\mathbf{b} = (1, 1)^T$. What is the condition number of A? What is $\mathbf{x}$?

b) Write a Matlab script to repeat the above calculation 50 times, each time perturbing $\mathbf{b}$ with 1 percent Gaussian noise. Generate a scatter plot of all fifty solutions $\mathbf{x}_i$. Form a histogram of the quantities $\|\mathbf{x}_i - \mathbf{x}\|$.

c) Repeat both items above with the same $\mathbf{b}$, but with the matrix A replaced by $\begin{bmatrix} 0.16 & -0.23 \\ -0.23 & 0.94 \end{bmatrix}$.

d) Is there any notable difference in the two histograms? How do you explain this difference?