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The deterministic EPQ with partial backordering: A new approach☆

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Abstract

Several authors have developed models for the EOQ when only a percentage of stockouts will be backordered. Most of these models are complicated, with equations unlike those for the EOQ with full backordering. In this paper we extend work by Pentico and Drake [The deterministic EOQ with partial backordering: a new approach. European Journal of Operational Research 2008; in press] that developed equations for the EOQ with partial backordering that are more like those for the EOQ with full backordering to develop a comparable model for the EPQ with partial backordering.

Keywords: Inventory control; Production lot sizing; Partial backordering

1. Introduction

The two basic questions any inventory control system must answer are when and how much to order. Over the years, hundreds of papers and books have been published presenting models for doing this under various conditions and assumptions. The best known of these models is Harris's [2] classic square root economic order quantity (EOQ) model that appears in every basic textbook covering inventory management. While this model has been criticized for the unreasonableness of its assumptions (see, e.g., [3]), surveys have shown that it is widely used. Further, it forms the basis for many other models that relax one or more of its assumptions. See Jagieta and Michenzi [4] and Khouja and Park [5] for examples of extensions to the classic EOQ model that relax one or more of its traditional assumptions.

An early extension of the basic EOQ model, generally referred to as the economic *production* quantity (EPQ) or economic *manufacturing* quantity (EMQ) model, replaced the assumption of instantaneous replenishment by the assumption that the replenishment order is received at a constant finite rate over time.

A key assumption of both the basic EOQ and EPQ models is that stockouts are not permitted. Assuming that the lead time and demand are known and constant, this means that an order will be placed when the inventory available is exactly sufficient to cover the demand during that lead time. Under conditions of demand certainty, however, it is possible to prove that, assuming customers are always willing, although not necessarily happy, to wait for delivery, planned backorders can make economic sense, even if they incur some actual or implied cost. Relaxing the basic EOQ and EPQ models' assumption that stockouts are not permitted led to the development of both EOQ and EPQ models for the two

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basic stockout cases: backorders and lost sales. What took longer to develop was a model that recognized that, while some customers are willing to wait for delivery, others are not. Either these customers will cancel their orders or the supplier will have to fill them within the normal delivery time by using more expensive supply methods.

While there have been a number of models developed for the EOQ and EPQ with partial backordering, most of them incorporate considerably more complicated assumption sets than the classic EOQ and EPQ models do. After reviewing the models for the "pure" stockout cases of backorders and lost sales, we will briefly discuss five models for the basic EOQ with partial backordering and will summarize the only paper we are aware of that develops a model for the basic EPQ problem with partial backordering. Following that, we will present an alternative approach to modeling the latter problem and determining expressions for when and how much to order.

2. Notation and terminology

A significant difficulty in reading the literature in inventory modeling is that there is no standard set of notation. We will use notation that, in our opinion, makes it somewhat easier to remember what the different symbols represent.

Parameters

D	demand per year
P	production rate per year if constantly producing
s	the unit selling price
C_o	the fixed cost of placing and receiving an order
C_p	the variable cost of a purchasing or producing a unit
C_h	the cost to hold a unit in inventory for a year
C_b	the cost to keep a unit backordered for a year
C_g	the goodwill loss on a unit of unfilled demand
$C_l = (s - C_p) + C_g$	the cost for a lost sale, including the lost profit on that unit and any goodwill loss
β	the fraction of stockouts that will be backordered

Variables

Q	the order quantity
T	the length of an order cycle
I	the maximum inventory level, with $\bar{I}$ being the average inventory level over the year
S	the maximum stockout level, including both backorders and lost sales
B	the maximum backorder position, with $\bar{B}$ being the average backorder level over the year ($B = \beta S$)
F	the fill rate or the percentage of demand that will be filled from stock

3. The "pure" stockout cases: backorders and lost sales

Models for the EOQ and EPQ with backorders appear in many basic texts. While the analysis for lost sales appears less frequently, Zipkin [6] and Pentico and Drake [1] give derivations for the basic EOQ with lost sales that are easily generalized to the EPQ case.

3.1. The "pure" backorder models

Since many basic texts derive models for the EOQ and EPQ with full backordering, we will not go through those derivations. There have also been many extensions of the full backordering EOQ to scenarios that relax other basic EOQ assumptions as well; see Wee et al. [7] for an example of an EOQ model with defective units and full backordering. Using the notation in the previous section, the relevant equations for the optimal order quantity (Q^*), maximum backorder quantity (B^*), and time between orders (T^*) are as follows:

3.1.1. EOQ with full backordering

$$Q^* = \sqrt{\frac{2C_oD}{C_h}}\sqrt{\frac{C_b + C_h}{C_b}},$$

$$B^* = Q^*\left(\frac{C_h}{C_b + C_h}\right) \quad \text{and}$$

$$T^* = \sqrt{\frac{2C_o}{DC_h}}\sqrt{\frac{C_b + C_h}{C_b}}. \tag{1}$$

3.1.2. EPQ with full backordering

Fig. 1 is a graph of the net inventory level for the EPQ with full backordering. The equations for the optimal order quantity (Q^*) and time between orders (T^*) for the EPQ with full backordering problem are almost

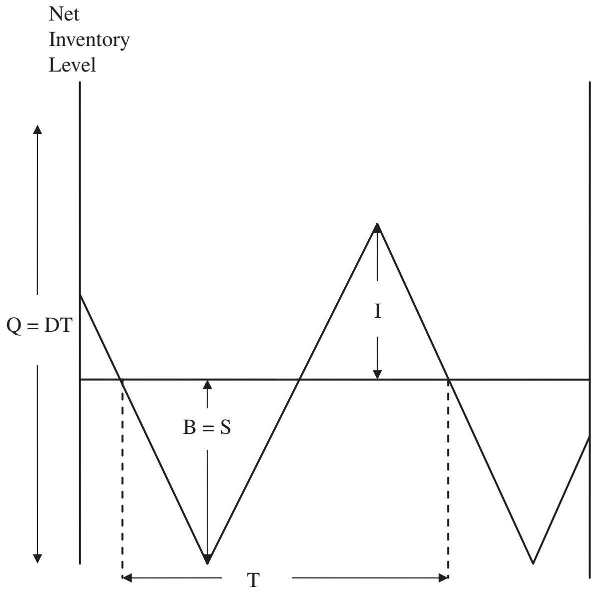

Fig. 1. Graph of pure backorder case for EPQ.

identical to those for the EOQ with full backordering, the difference being the multiplication of C_h in the first square root of both Q^* and T^* by $(1-D/P)$, reflecting the relative sizes of the production and usage rates:

$$Q^* = \sqrt{\frac{2C_oD}{C_h(1-D/P)}}\sqrt{\frac{C_b+C_h}{C_b}} \quad \text{and}$$

$$T^* = \sqrt{\frac{2C_o}{DC_h(1-D/P)}}\sqrt{\frac{C_b+C_h}{C_b}}. \tag{2}$$

The equation for B^* for the EPQ differs from the comparable equation for the EOQ in a different way, but is still involves adjusting for the relative sizes of the production and usage rates:

$$B^* = \sqrt{\frac{2C_oD}{C_b}}\sqrt{\frac{C_h}{C_b+C_h}}\sqrt{1-\frac{D}{P}}. \tag{3}$$

3.2. The "pure" lost sales case

Zipkin [6] and Pentico and Drake [1] show, for the EOQ problem, that if demands during a stockout period are lost sales rather than resulting in backorders, the optimal policy is to have either no stockouts or all stockouts, the choice being the alternative with the lower average cost per period. Zipkin's proof is actually for the situation in which stockouts are backordered at a fixed cost per unit rather than incurring a cost proportional to the amount of time that the unit is backordered. He then develops the cost equation for the lost sales case and, showing that it has the same form as the "backorder incidence" case, concludes that the same result holds. Pentico and Drake prove the same result directly, using the notation and decision variables used in this paper. It is straightforward to show that the same result holds for the EPQ with lost sales.

4. Previous research on models with partial backordering

Since, for the EOQ model, it is optimal to allow some stockouts if all customers will wait ($\beta=1$) and to either allow no stockouts or lose all sales if no customers will wait ($\beta=0$), it is logical to assume there will be a value of β below which one should use the optimal ordering policy for the lost sales case and above which one should allow stockouts, some of which will be backordered. This is, in fact, what happens. Determining an optimal policy for the partial backordering deterministic EOQ problem starts with identifying the minimum value of β for which stockouts should be allowed and, if β is greater than this minimum value, determining the optimal order quantity.

We begin this brief survey with a discussion of models that are identical to the basic EOQ model except for allowing for partial backordering. Models for the partial backordering EOQ problem were developed by Montgomery et al. [8], Rosenberg [9], Park [10], San José et al. [11–14], and Pentico and Drake [1]. These papers took somewhat different approaches to modeling the problem, differing in some cases in their cost assumptions, but primarily in which decision variables they used.

Both Montgomery et al. [8] and Rosenberg [9] include a fixed backorder charge, which is the same as the goodwill loss for a lost sale. They solve for the optimal ordering quantity through a two-step optimization procedure since the overall cost function is not necessarily convex; the difference in their models is due to the decision variables used. Montgomery et al. use the order quantity, Q, and the maximum backorder level, S, but transform them into two new variables: $U=Q+(1-\beta)S$, which is the total amount of demand during an order cycle, or TD in our notation, and $V=Q-\beta S$, which, in our notation, is I, the on-hand inventory at the beginning of the cycle. Rosenberg also begins with Q and S as his decision variables but replaces them with two other variables: T, the length of the inventory review cycle, given by $T=[Q+(1-\beta)S]/D$, and a fictitious demand rate X, defined as $X=(Q-\beta S)/T$.

Park [10] does not include a fixed unit cost per unit backordered. His choice of variables—S, the maximum size of the stockout during an inventory cycle, and R, which is the same as U in Montgomery et al.—enables him to show that his cost function is convex, so he can develop a solution by simultaneously solving the two equations developed by setting the partial derivatives equal to 0. Determining the solution involves evaluating a complicated expression for S^* as the solution to a quadratic equation. From the equation for S^* he develops a statement of the range of values for β for which S^* would have to be 0, in which case either all sales are lost or no stockouts are allowed and Q^* is determined with the basic EOQ model.

The cost structure used by San José et al. [11–14] also includes a fixed unit cost of backordering, but it is not necessarily the same as the goodwill loss for a lost sale. More significantly, in all of their models except the one presented in [11], they assume that the percentage of unmet demand backordered is not a constant. Along with analyzing the constant-β case, they consider a variety of different "customer impatience" functions in which the percentage backordered increases as the replenishment date approaches. San José et al.'s decision variables are T, the length of time in an inventory cycle during which inventory is positive (FT in our notation), and Ψ, the complementary time during which there are backorders ($(1-F)T$ in our notation). They replace these two variables by Ψ and $\alpha = T + \Psi$, which is the same as our T. As in Montgomery et al. [8] and Rosenberg [9], their solution procedure is executed in two stages, first finding an optimal value for Ψ and then substituting that into an equation that relates the value of α to the value of Ψ.

Pentico and Drake's [1] objective was to develop a set of equations that are both simpler to use and have a more understandable and intuitive form that more closely resemble the comparable equations for the basic EOQ and EOQ with full backordering. Because their approach forms the basis for what we will do here, we will cover it in greater detail than those of the previous authors.

As in Park [10], Pentico and Drake did not include a fixed unit cost to backorder, although they did consider that case in an Appendix in [1]. Their decision variables are T, the length of an order cycle, which is the same as San José et al.'s α, and F, the fraction of demand to be filled from stock. Setting the derivatives of their objective function with respect to T and F equal to 0 gives an expression for T as a function of F

$$T(F) = \sqrt{\frac{2C_o}{D[C_h F^2 + \beta C_b (1-F)^2]}} \tag{4}$$

and an equation for F as a function of T

$$F(T) = \frac{(1-\beta)C_l + \beta C_b T}{T(C_h + \beta C_b)}. \tag{5}$$

Substituting the expression for F in (5) into Eq. (4) leads to:

$$T^* = \sqrt{\frac{2C_o}{DC_h}\left[\frac{C_h + \beta C_b}{\beta C_b}\right] - \frac{[(1-\beta)C_l]^2}{\beta C_h C_b}}. \tag{6}$$

Recognizing that T^* for partial backordering must be at least as large as T^* for the basic EOQ ($\sqrt{2C_o/(DC_h)}$) if partial backordering is optimal leads to the following bound on β:

$$\beta \geqslant \beta^* = 1 - \sqrt{\frac{2C_o C_h}{DC_l^2}}. \tag{7}$$

This is the same condition derived by Park [10]. An alternative way of stating this condition on β, which makes it comparable to the condition given in Rosenberg [9], which is equivalent to the one in Montgomery et al. [8], is

$$\sqrt{\frac{2C_o}{DC_h}} \geqslant \frac{(1-\beta)C_l}{C_h}. \tag{8}$$

In addition to models based on the straightforward partial-backordering generalization of the EOQ model discussed above, there have been a variety of additional studies that model the partial backordering behavior in more complicated decision environments. Some of these papers consider deteriorating inventory, items such as fresh produce and semiconductor chips that lose their value quickly from damage, obsolescence, or pilferage. Some include time-varying demand and/or pricing decisions. Abad [15] models the joint dynamic pricing and ordering decisions with perishable inventory and time-varying backorder percentages defined by customer impatience functions similar to some of those used by San José et al. [11,13]. He investigates the computational properties of this model when the unit selling price is fixed within the inventory cycle in [16]. Neither of these models incorporates expressions for stockout or backordering costs, since the author claims that these parameters are difficult to estimate in practice. Dye [17] extends the model in [16] by including the stockout and backorder costs. Wee [18] considers the same pricing and ordering decisions for perishable items under linear demand with quantity discounting and a constant backordering percentage. Chang and Dye [19] develop an optimal ordering model for time-varying demand and

Abad's [15] backordering impatience function. An additional model utilizing partial backordering is Yang et al. [20], which considers optimal lot sizes for integrated supply chains under both perfect and monopolistic competition.

In addition to these papers, and others, that consider partial backordering in the context of batch ordering, there are a number of papers that consider the combination of partial backordering and a finite production rate, which is the problem that we are addressing in this research. As is the case with most of the above-referenced papers, most of these authors also generally looked at more complex decision environments. Abad [21] extended his previous work based on the EOQ to consider the combination of perishable goods and a finite production rate. Unlike his approach in [15,16], in [21] Abad included costs for both backordering and lost sales and used a constant backordering percentage. Goyal and Giri [22] allow the demand, production, and deterioration rates to vary over time. Giri et al. [23] develop a model for the case in which demand is increasing, the production rate is finite and can be adjusted for each production cycle, and shortages will be partially backordered. In addition to a time-varying rate of inventory deterioration, Lo et al. [24] include inflation, an imperfect production process, and multiple deliveries in their situational scenario. Jolai et al. [25] included perishable inventory, a constant backordering percentage, demand that decreases linearly with the decreasing inventory level, a finite production rate, inflation, and, unlike the other papers referenced here, a finite planning horizon. They assume that each production/demand cycle will be of the same length and solve for m, the integer number cycles during the finite planning horizon, and T_2, the length of time from when inventory reaches 0 until the next production run begins.

5. Modeling net inventory during the stockout period for the EPQ with backordering

The only paper we have found that develops a model for the basic EPQ with partial backordering under basically the same assumptions as we consider in this paper is by Mak [26]. Li et al. [27] discuss an EPQ model for multiple products with planned backorders, but they assume that all of the demand not originally satisfied from stock is willing to wait for the units to be delivered upon production. Chakraborty et al. [28] develop a model for the EPQ that simultaneously considers the effect of production defects and machine breakdowns; in their model, however, all shortages result in lost sales. Since Mak's approach to the treatment of demands that

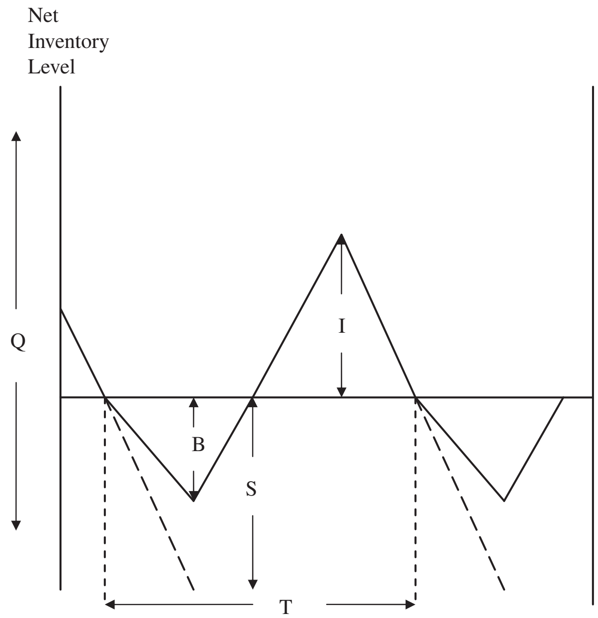

Fig. 2. Graph of partial backorder case for EPQ with LIFO backorder filling (adapted from [26]).

occur while there is no stock but the production run has started differs from our approach, we first discuss that issue.

From the time the system runs out of stock until the time the next order is received (EOQ with backordering) or the next production run begins (EPQ with backordering), a fraction β of incoming demand will be backordered until the maximum backorder level $B = \beta S$ is reached. In the EOQ with full or partial backordering, the entire order quantity Q is received simultaneously, so all the backorders can be filled at once, with the inventory rising immediately to $I = Q - B$. In the EPQ with full backordering, the order quantity Q is received in a constant stream at a rate of P. Since all demands that occur during the time it takes to fill all the backorders are also backordered if they are not filled immediately, it makes no difference whether the incoming orders are filled before the backorders (a LIFO approach) or the backorders are filled before the incoming orders (a FIFO approach). A graph of the net inventory level for the EPQ with full backordering is shown in Fig. 1.

For the EPQ with partial backordering; however, whether LIFO or FIFO is used to determine the order in which new demands and backorders are filled after the production run begins can make a difference in the net inventory level. The impact depends on the answer to an additional question: What happens to the demands that occur when there is no stock on hand but the production run has been started? If we assume

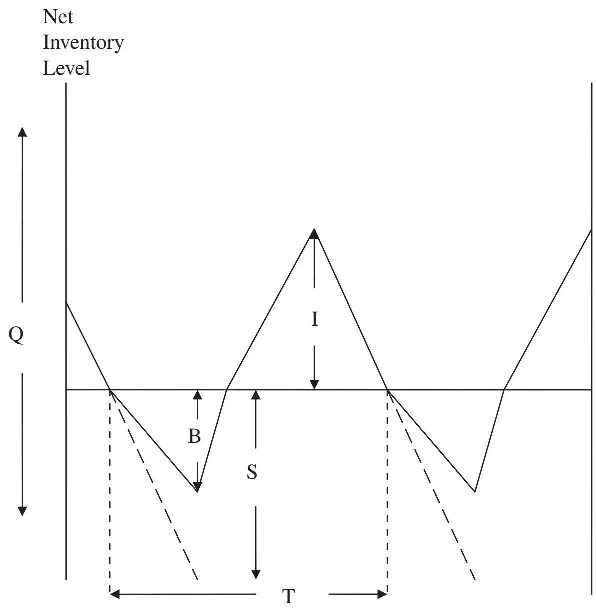

Fig. 3. Graph of partial backorder case for EPQ with FIFO backorder filling.

that incoming demands will be filled from production before existing backorders are filled (LIFO) and that none of the existing backorders will convert to lost sales, then the net inventory level for the EPQ with partial backordering will be as shown in Fig. 2. If, however, we assume that the existing backorders will be filled before any new demands are filled (FIFO) and assume that only a fraction β of these new orders that cannot immediately be filled will be backordered, with the rest being lost sales, then the net inventory level for the EPQ with partial backordering will be as shown in Fig. 3. (If all incoming orders will wait once the production run has started, it makes no difference whether LIFO or FIFO is used and Fig. 2 applies.)

6. Mak's [26] model for the EPQ with partial backordering

Mak's assumptions are the usual ones for the basic EPQ with full backordering except that, as in all the basic EOQ with partial backordering models summarized above except the ones by San José et al. [11–14], only a constant fraction β of the stockouts will be backordered, with the rest being lost sales. Relative to the discussion immediately above about whether LIFO or FIFO is used to fill the backorders once the production run starts, Mak's approach is as shown in Fig. 2. He assumes that there will be no increase in either backorders or lost sales once the production phase begins so the backorders are filled at a rate of $P - D$. A model based on Mak's LIFO assumption that uses our decision variables is developed in Appendix C.

Mak's decision variables are T, the length of an inventory cycle, and t, the length of time from when the inventory level reaches 0 until the next production run begins. The cost function he develops is convex, and thus the optimal solution can be found by setting the two partial derivatives equal to 0 and solving the resulting equations simultaneously. He does this by developing an equation for T as a function of t and then, by using this to eliminate T from the other equation, finds an expression for t^* as a function of the parameters and, using this, finds an expression for T^*. Both of these equations are quite complicated. Mak also develops a statement of the condition that β must satisfy for the partial backordering EPQ equations to apply that is comparable to the ones developed for the partial backordering EOQ models and the one to be developed here.

7. An alternative approach to the EPQ with partial backordering

We use the same assumptions about costs and demand as used in the basic EOQ with full backordering model and by Mak [26]. However, we assume that a FIFO policy is used to fill the backorders once the production run starts, so that Fig. 3 shows the level of net inventory over the course of an inventory cycle. As in Pentico and Drake [1], we use T, the length of an inventory cycle, and F, the fill rate, as the decision variables. Our objective, as in Pentico and Drake, is to develop a set of equations that are simpler to use and have a more understandable form than those in Mak.

7.1. Time intervals and the maximum inventory and backorder levels

As shown in Fig. 4, an inventory cycle, which has length T, can be divided into four sub-intervals. As developed in Appendix A, the lengths of these intervals are

$$t_1 = (1 - F)T(1 - \beta D/P),$$
$$t_2 = (1 - F)T(\beta D/P),$$
$$t_3 = FTD/P,$$
$$t_4 = FT(1 - D/P).$$

Using these values we develop expressions for I, the maximum inventory level, and B, the maximum backorder level.

- As in the EPQ with full backordering model, during interval 3 the inventory level is increasing at a rate of

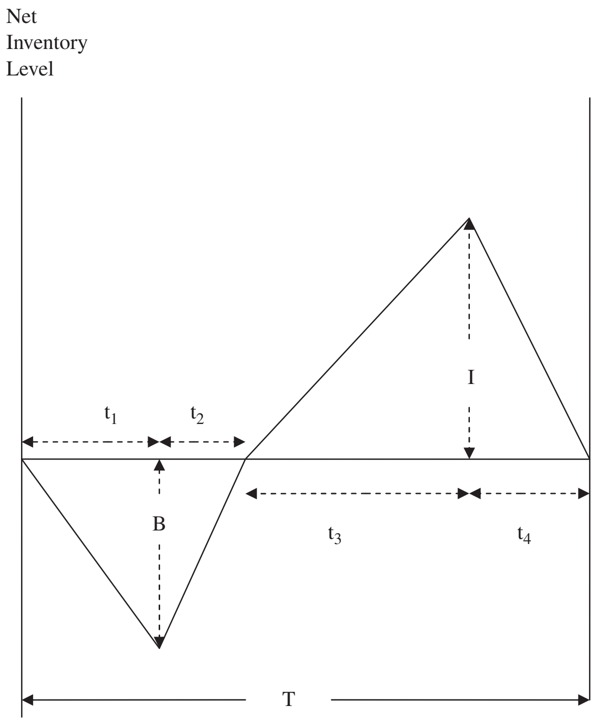

Fig. 4. Graph showing time intervals for EPQ with partial backordering and FIFO backorder filling.

$P-D$, so $I=(P-D)t_3$. Substituting the expression for t_3 above, we get $I=FTD(1-D/P)$.

- During interval 1, no production is taking place and no demands are filled, so the backorder grows from 0 to its maximum value, $B=\beta Dt_1$. Substituting the expression for t_1 above, we get $B=\beta D(1-F)T(1-\beta D/P)$.

7.2. The profit and cost functions based on T and F

The average profit per year to be maximized is the revenue from filling demands, either from stock or as backorders, minus the cost of placing orders, the cost of the units used or sold, the cost of carrying inventory, the cost of the backorders, and the cost of lost sales. Thus,

$$\begin{aligned}\Pi(T,F) &= (s-C_p)D[F+\beta(1-F)] \\ &\quad -[C_o/T+C_h\bar{I}+C_b\bar{B} \\ &\quad +C_gD(1-\beta)(1-F)] \\ &= (s-C_p)D-\Gamma(T,F), \end{aligned} \tag{9}$$

where

$$\begin{aligned}\Gamma(T,F) &= C_o/T+C_h\bar{I}+C_b\bar{B} \\ &\quad +C_lD(1-\beta)(1-F). \end{aligned} \tag{10}$$

Since $(s-C_p)D$ is a constant, $\Pi(T,F)$ is maximized by the pair (T,F) that minimizes $\Gamma(T,F)$.

From Fig. 4, we see that the average inventory is given by the average value of a triangle with height I and base $t_3+t_4=FT$, multiplied by the fraction of time for which there is inventory, F. Since $I=FTD(1-D/P)$, the average inventory is

$$\bar{I}=\frac{DTF^2}{2}\left(1-\frac{D}{P}\right). \tag{11}$$

Also from Fig. 4, we see that the average backorder level is given by the average value of a triangle with height B and base $t_1+t_2=(1-F)T$, multiplied by the fraction of time for which there are backorders, $1-F$. Since $B=\beta D(1-F)T(1-\beta D/P)$, the average backorder level is

$$\bar{B}=\frac{\beta DT(1-F)^2}{2}\left(1-\frac{\beta D}{P}\right). \tag{12}$$

Substituting the expressions for $\bar{I}$ in (11) and $\bar{B}$ in (12) into (10) gives

$$\begin{aligned}\Gamma(T,F) &= \frac{C_o}{T}+\frac{C_hDTF^2}{2}\left(1-\frac{D}{P}\right) \\ &\quad +\frac{\beta C_bDT(1-F)^2}{2}\left(1-\frac{\beta D}{P}\right) \\ &\quad +C_lD(1-\beta)(1-F). \end{aligned} \tag{13}$$

To simplify the notation, we define $C_h'=C_h(1-D/P)$ and $C_b'=C_b(1-\beta D/P)$, which makes the average cost per year to be minimized:

$$\begin{aligned}\Gamma(T,F) &= \frac{C_o}{T}+\frac{C_h'DTF^2}{2}+\frac{\beta C_b'DT(1-F)^2}{2} \\ &\quad +C_lD(1-\beta)(1-F). \end{aligned} \tag{14}$$

7.3. Determining the optimal values for T and F

Taking the partial derivative of $\Gamma(T,F)$ in (14) with respect to T and setting it equal to 0 gives

$$\frac{\partial\Gamma}{\partial T}=-\frac{C_o}{T^2}+\frac{C_h'DF^2}{2}+\frac{\beta C_b'D(1-F)^2}{2}=0. \tag{15}$$

This gives, after some algebra:

$$T=\sqrt{\frac{2C_o}{D[C_h'F^2+\beta C_b'(1-F)^2]}}. \tag{16}$$

Note that this equation for T has the same general form as the equation for the optimal T for the EPQ with full backordering model given in (2) and, in fact, reduces to that equation if $\beta=1$. Further, this equation for T also reduces exactly to the equation for the basic

EPQ model if $F=1$, which means that there will be no stockouts.

Taking the partial derivative of $\Gamma(T, F)$ with respect to F and setting it equal to 0 gives

$$\frac{\partial \Gamma}{\partial F} = C_h' DTF - \beta C_b' DT(1-F) - (1-\beta)C_l D = 0. \tag{17}$$

After some algebra, this results in

$$F(T) = \frac{(1-\beta)C_l + \beta C_b' T}{T(C_h' + \beta C_b')}. \tag{18}$$

Substituting this expression for F into Eq. (16), we get, after some algebra:

$$T^* = \sqrt{\frac{2C_o}{DC_h'}\left[\frac{C_h' + \beta C_b'}{\beta C_b'}\right] - \frac{[(1-\beta)C_l]^2}{\beta C_h' C_b'}}. \tag{19}$$

Using the value of T^* from (19) in (18) gives $F(T^*)$ and completes the solution.

Substituting the formulas for T^* in (19) and $F(T^*)$ in (18) into the equation for $\Gamma(T, F)$ in (14) gives, after a considerable amount of algebra, a simple expression for the optimal cost for the EPQ with partial backordering:

$$\Gamma^* = \Gamma(T^*, F^*) = C_h' DT^* F^*. \tag{20}$$

It is interesting to note that Γ^* for this model, as given in (20), has exactly the same form as it does for the comparable models for the basic EOQ, the EOQ with full backordering, and the EOQ with partial backordering and constant β, for all of which $\Gamma^* = C_h F T^* F^*$, and for the models for the basic EPQ and the EPQ with full backordering, for both of which $\Gamma^* = C_h' DT^* F^*$, where $C_h'(1-D/P)$.

Recognizing that T^* for the partial backordering model must be at least as large as T^* for the basic EPQ ($\sqrt{2C_o/(DC_h')}$) if backordering is optimal gives the bound

$$\frac{2C_o}{DC_h'}\left[\frac{C_h' + \beta C_b'}{\beta C_b'}\right] - \frac{[(1-\beta)C_l]^2}{\beta C_h' C_b'} \geqslant \frac{2C_o}{DC_h'}.$$

After some algebra, this leads us to the following conclusion: For the equations for T^* and F^* to give an optimal solution, we must have

$$\beta \geqslant \beta^* = 1 - \sqrt{\frac{2C_o C_h'}{DC_l^2}}. \tag{21}$$

With a little algebra, an alternate form of this condition on β is

$$\sqrt{\frac{2C_o}{DC_h'}} \geqslant \frac{(1-\beta)C_l}{C_h'}. \tag{22}$$

In Appendix B we prove that the equations in (19) and (18) give an optimal solution for the EPQ with partial backordering and constant β if the condition on β in (21) or (22) is met and a FIFO policy on filling backorders is followed.

It is encouraging to note that the form of the equation for T^* in (19) is very similar to the equation for T^* for the full backordering case given in (2): Determining T^* begins with multiplying the value of T^* for the basic EPQ model by a term that reflects the relative sizes of the unit inventory cost per year and the unit backorder cost per year, although the backordering cost component has been multiplied by β to reflect the fact that only a percentage of the stockouts will be backordered. This initial value of T^* is then reduced by a term that reflects the relative cost of having a unit of demand result in a lost sale to the cost of having that unit of demand eventually satisfied, either from inventory or by being backordered. Similarly, the equation for F^* in (18) is logical in that it reflects the relative sizes of the cost of not filling a unit of demand from stock and the cost of filling a unit of demand, whether immediately from stock or eventually by being backordered.

It is also interesting to note that, with the replacement of C_h' for C_h and C_b' for C_b, the equations for T^* and F^* for the EPQ with partial backordering given in (19) and (18) and the condition for determining the minimum value of β for which backordering is optimal given in (21) and (22) are identical to the equations for T^* and F^* for the EOQ with partial backordering in Pentico and Drake [1], reproduced here in (6) and (5) and the condition on β reproduced in (7) and (8).

The procedure for determining the optimal values for T, F, Q, I, S, and B is, then:

1. Determine β^*, the critical value for β, from (21).
2. (a) If $\beta \leqslant \beta^*$, determine T^* from the basic EPQ model ($T^* = \sqrt{2C_o/(DC_h(1-D/P))}$) and determine the optimal cost of allowing no stockouts ($\Gamma^* = \sqrt{2C_o C_h D(1-D/P)}$). Compare this with the cost of losing all demand, $C_l D$, to determine whether it is optimal to allow no stockouts or all stockouts.
   (b) If $\beta > \beta^*$, use (19) to determine the value of T^* and substitute it into (18) to determine the value of F^*.

3. Determine the values of the other variables and cost as follows:
Total demand during a cycle $= DT^*$.
Maximum inventory $= I^* = F^* DT^*(1 - D/P)$.
Maximum stockout $= S^* = Dt_1^* = (1 - F^*)T^*D(1 - \beta D/P)$.
Maximum backorder $= B^* = \beta S^*$.
Order quantity $= Q^* = DT^*[F^* + \beta(1 - F^*)]$.
Average total cost per period $= C_h' DT^* F^*$.

8. Numerical example

To illustrate the procedure, we will use the numerical example from Mak [26]. However, the value of β that Mak uses ($\beta = 0.75$), while acceptable with a LIFO policy, does not meet the criterion for the optimality of partial backordering under a FIFO inventory policy. As a result, we use a different value of β and, thus, obtain different results. The remainder of the parameter values for Mak's example are

$D = 1100$ units per year,
$P = 9200$ per year,
$C_o = \$275$ per setup,
$C_h = \$2.00$ per unit per year,
$C_b = \$3.20$ per unit per year,
$C_l = \$4.00$ per unit.

The first step is to determine the values of C_h' and C_b':

$$C_h' = C_h(1 - D/P) = 2.00(1 - 1100/9200) = 2.00(0.880435) = 1.761.$$

$$C_b' = C_b(1 - \beta D/P) = 3.20(1 - 0.119565\beta),$$

which cannot be evaluated further until we have a value for β.

1. From (20) determine

$$\beta^* = 1 - \sqrt{\frac{2C_o C_h'}{DC_l^2}} = 1 - \sqrt{\frac{2(275)(1.761)}{(1100)(16.00)}} = 1 - 0.2346 = 0.7654.$$

For $\beta = 0.50$:

2. Since $\beta < \beta^* = 0.7654$, compare the cost of the basic EPQ model with the cost of not stocking at all. Using the EPQ model:

$$T^* = \sqrt{2C_o/(DC_h(1 - D/P))} = \sqrt{2(275)/((1100)(2.00)(1 - 1100/9200))} = 0.53287.$$

$$\Gamma^* = \sqrt{2C_o C_h D(1 - D/P)} = \sqrt{2(275)(2.00)(1100)(1 - 1100/9200)} = 1032.15.$$

Since the cost of not stocking at all has an annual cost of $C_l D = (4.00)(1100) = 4400$, the optimal strategy for $\beta = 0.50$ is to use the basic EPQ model with $T^* = 0.53287$ and $Q^* = DT^* = (1100)(0.53287) = 586$.

For $\beta = 0.90$:

2. Since $\beta > \beta^* = 0.7654$, proceed to Step 3.
3. First, determine $C_b' = 3.20(1 - 0.1195655\beta) = 3.2(1 - (0.119565)(0.90)) = 3.20(1 - 0.1076) = 3.20(0.8924) = 2.8557$.

Using (19),

$$T^* = \sqrt{\frac{2C_o}{DC_h'}\left[\frac{C_h' + \beta C_b'}{\beta C_b'}\right] - \frac{[(1-\beta)C_l]^2}{\beta C_h' C_b'}} = \sqrt{\frac{2(275)}{(1100)(1.761)}\left[\frac{1.761 + (0.90)(2.8557)}{(0.90)(2.8557)}\right] - \frac{[(1-0.90)(4.00)]^2}{(0.90)(1.761)(2.8557)}}$$
$$= \sqrt{((0.28393)(1.6852) - 0.03535)} = \sqrt{0.4431} = 0.6657.$$

Using (18),

$$F^* = \frac{(1-\beta)C_l + \beta C_b' T^*}{T^*(C_h' + \beta C_b')} = \frac{(1 - 0.90)(4.00) + [(0.90)(2.8557)(0.6657)]}{(0.6657)(1.761 + (0.90)(2.8557))} = \frac{2.111}{2.883} = 0.732.$$

The values of the other decision variables are:

Total demand during a cycle $= DT^* = (1100)(0.6657) = 732.27$

Maximum inventory $= I^* = F^* DT^*(1 - D/P) = (0.732)(1100)(0.6657)(1 - 1100/9200) = 471.93.$

Maximum stockout $= S^* = (1 - F^*)DT^*(1 - \beta D/P) = (1 - 0.732)(1100)(0.6657)(0.8924) = 175.13.$

Maximum backorder $= B^* = \beta S^* = (0.90)(175.13) = 157.62.$

$$\begin{aligned}\text{Order quantity} &= Q^* = DT^*[F^* + \beta^*(1 - F^*)] \\ &= 732.27[0.732 + (0.90)(1 - 0.732)] \\ &= 732.27[0.732 + 0.2412] \\ &= (732.27)(0.9732) = 712.65.\end{aligned}$$

$$\begin{aligned}&\text{The average cost per period} = C_{\rm h}' DT^* F^* \\ &\quad = (1.761)(1100)(0.6657)(0.732) = 943.93.\end{aligned}$$

9. Conclusions and future work

As noted by previous authors in the context of the EOQ model, determining the optimal ordering and stockout quantities when demands that cannot be filled from stock are partially backordered is much more complicated than for the cases in which all stockouts are either backordered or result in lost sales. As shown here, the same is true for the EPQ model. However, by changing the decision variables from Q, the order quantity, and S, the stockout level, to T, the time between orders, and F, the fill rate, we have developed a model with equations that are more like those for the basic EPQ model and its full backordering extension and are much easier to solve than the equations developed by Mak [26].

There are several possible extensions to our EPQ model. The most obvious one is the relaxation of the assumption of a constant backordering rate. In many instances it is likely that a larger percentage of customers would be willing to wait for a backordered product as the start of the next production run gets closer in time. This can be captured by incorporating a time-dependent backordering rate. Several EOQ models, such as those of San Jose et al. [11–14], have been developed with this time-varying rate, but to our knowledge there are no EPQ models for this situation. Another possible extension is to include a fixed backordering charge in addition to or instead of our existing cost that depends on the length of the backorder.

Appendix A. Derivation of the subinterval lengths in Fig. 4

- Interval 1, with length t_1, extends from when the net inventory level first becomes 0 until it reaches the maximum backorder level, B. During this interval the backorder is accumulating at a rate of βD per year, so $B = \beta D t_1$.
- Production starts at the beginning of interval 2, which has length t_2. During this time the backorder level is being reduced. However, since demands continue to come in and cannot be filled until all the backorder has been satisfied, the net inventory increases (and the backorder decreases) at a rate of $P - \beta D$.
- Interval 3, which has length t_3, begins when the backorders have finally been eliminated and continues until the full order quantity, Q, has been produced and the inventory reaches its maximum level, I. As in the EPQ with full backordering model, the inventory level is increasing at a rate of $P - D$ during this interval, so $I = (P - D)t_3$.
- Interval 4, which has length t_4, begins when the inventory level reaches its maximum and production stops. During this interval the inventory level is decreasing at a rate of D.

With these considerations in mind, we can determine the values of t_1, t_2, t_3, and t_4 in terms of the parameters and decision variables.

- From the beginning of interval 3 to the end of interval 4, all demand is filled from stock, so $t_3 + t_4 = FT$ and the total amount demanded and produced during this time is FTD, all of which is produced during interval 3. Thus, $t_3 = FTD/P$. At the end of interval 3 the inventory level is I, which is the amount produced during interval 3 minus the amount demanded during interval 3. Thus $I = FTD - t_3 D = D(FT - FTD/P) = FTD(1 - D/P)$. Since $t_4 = FT - t_3$, $t_4 = FT - FTD/P = FT(1 - D/P)$.
- Intervals 1 and 2 complement intervals 3 and 4, so $t_1 + t_2 = (1 - F)T$. During interval 1, no production is taking place and no demands are filled, so, as noted above, the backorder grows from 0 to its maximum value, $B = \beta D t_1$.
- During interval 2, this backorder will be eliminated. Assuming a FIFO policy on filling demand, the time required to eliminate the initial backorder is $\beta D t_1 / P$. During this time, additional demands will be coming in, a fraction β of which will be backordered, and the time to eliminate this second amount of backorder is $\beta D(\beta D t_1/P)/P = (\beta D/P)^2 t_1$. Similarly, additional demands will be coming in during this time, a fraction β of which will be backordered, and the time to eliminate this third amount of backorder is $\beta D((\beta D/P)^2 t_1)/P = (\beta D/P)^3 t_1$. Continuing in this fashion, we find that

$$\begin{aligned}t_2 &= t_1 \left[\frac{\beta D}{P} + \left(\frac{\beta D}{P}\right)^2 + \left(\frac{\beta D}{P}\right)^3 + \left(\frac{\beta D}{P}\right)^4 + \cdots\right] \\ &= t_1 \left[\frac{\beta D/P}{1 - \beta D/P}\right].\end{aligned}$$

- Thus,

$$t_1+t_2=t_1\left[1+\frac{\beta D/P}{1-\beta D/P}\right]=(1-F)T,$$

which gives $t_1=(1-F)T(1-\beta D/P)$ and $t_2=(1-F)T(\beta D/P)$.

Appendix B. Proof of the optimality of the solution in (19) and (18) if (21) and (22) hold

Although the cost function in (14) is not convex, we can prove that the solution given by simultaneously solving (19) and (18) is a global optimum if β meets the condition in (21) or (22) by examining the characteristics of the partial derivatives and the boundary conditions.

The cost function in (14),

$$\Gamma(T,F)=\frac{C'_{\rm o}}{T}+\frac{C'_{\rm h}DTF^2}{2}+\frac{\beta C'_{\rm b}DT(1-F)^2}{2}+C_1D(1-\beta)(1-F),$$

can be rewritten as

$$\Gamma(T,F)=\frac{G_0}{T}+T(G_1F^2-2G_2F+G_2)-G_3F+G_3, \quad \text{(B1)}$$

where

$G_0=C_{\rm o}$,

$G_1=D(C'_{\rm h}+\beta C'_{\rm b})/2$,

$G_2=D\beta C'_{\rm b}/2$,

$G_3=C_1D(1-\beta)$.

Note that all the G_is are positive and $G_1>G_2$.

For ease of notation, we will rewrite (B1) as

$$\Gamma(T,F)=\frac{G_0}{T}+Tr(F)+q(F), \quad \text{(B2)}$$

where

$$r(F)=G_1F^2-2G_2F+G_2 \quad \text{(B3)}$$

and

$$q(F)=-G_3F+G_3. \quad \text{(B4)}$$

Our objective is to establish the conditions under which Eq. (B2) has a unique interior minimizer. Differentiating (B2) with respect to T yields

$$\frac{\partial\Gamma}{\partial T}=-\frac{G_{\rm o}}{T^2}+r(F),$$

which equals zero if and only if T satisfies

$$T^*=T^*(F)=\sqrt{\frac{G_0}{r(F)}}. \quad \text{(B5)}$$

Note that this is the same result, with appropriate changes of notation, given in (16).

Since the discriminant of $r(F)$ is negative, $r(F)$ has no roots. Thus, $r(F)$ is either all positive or all negative on its entire domain. Since $r(0)=G_2>0$, $r(F)$ is strictly positive in [0,1]. Thus, (B5) gives, for each F, a unique $T^*=T^*(F)$ that minimizes the cost function given by (B2).

Substituting the expression for $T^*(F)$ in (B5) into $\Gamma(T,F)$ given by (B2) gives

$$\hat{\Gamma}(F):\equiv\Gamma(T^*(F),F)=2\sqrt{G_0r(F)}+q(F), \quad \text{(B6)}$$

which represents that minimal possible cost for each value of F.

Note that $\hat{\Gamma}(F)$ is continuous, so on the compact interval [0,1] it has one or more local minima, the smallest of which will be the global minimum of the cost function. To find these minima, take the first and second derivatives of $\hat{\Gamma}(F)$ with respect to F, yielding

$$\hat{\Gamma}'(F)=\sqrt{G_0}\frac{r'(F)}{r(F)^{1/2}}+q'(F), \quad \text{(B7)}$$

$$\hat{\Gamma}''(F)=\frac{\sqrt{G_0}[2r''(F)r(F)-(r'(F))^2]}{2r(F)^{3/2}}, \quad \text{(B8)}$$

respectively. Note that $\hat{\Gamma}'(F)$, which is, with the change in notation, the same as $\partial\Gamma(T,F)/\partial F$ as given in (17), is continuous and satisfies $\hat{\Gamma}'(0)<0$ (since $r'(0)=-2G_2<0$, $r(0)=G_2>0$, and $q'(0)=-G_3<0$). Moreover:

$$\begin{aligned}\hat{\Gamma}'(1)&=\sqrt{G_0}\frac{r'(1)}{r(1)^{1/2}}+q'(1)\\&=\sqrt{G_0}\frac{2(G_1-G_2)}{\sqrt{G_1-G_2}}-G_3\\&=2\sqrt{G_0(G_1-G_2)}-G_3\\&=\sqrt{2C_0DC'_{\rm h}}-C_1D(1-\beta),\end{aligned}$$

which means that $\hat{\Gamma}'(1)>0$ if and only if

$$\sqrt{2C_{\rm o}/DC'_{\rm h}}>C_1(1-\beta)/C'_{\rm h}. \quad \text{(B9)}$$

Note that this is the strict inequality part of the condition on β given in (22). Finally, the second derivative $\hat{\Gamma}''(F)$

given in (B8) factors into

$$\hat{\Gamma}''(F) = \frac{2G_2\sqrt{G_0}}{r(F)^{3/2}}(G_1 - G_2),$$

which is positive for all F.

It now follows from elementary calculus that, if (B9) holds, then $\hat{\Gamma}(F)$ has a unique minimizer in the open interval (0,1); while if (B9) does not hold, the minimizer will lie on the boundary point $F = 1$. Note further that if $\hat{\Gamma}'(1) = 0$, which is the equality part of the condition on β given in (22), then the solution given by (19) and (18) is identical to the solution at the boundary point $F = 1$, which is the basic EPQ solution. Thus, if the condition on β given by (21) or (22) holds, the solution given by (19) and (18) minimizes the cost function given by (14).

Appendix C. The optimal policy for LIFO filling of backorders

The basic difference between the model developed in this paper and the one in Mak [26] is the assumption about how backorders are filled during Interval 2 in the order cycle, when the backorders have reached their maximum level B and production starts. As discussed and shown in Fig. 3, our model assumes a FIFO policy on filling the backorders. That is, the backorders are filled before the new demands. As shown in Fig. 2, Mak assumes (1) a LIFO policy in which incoming orders are filled first and the backorders are only filled from the excess production, and (2) that none of those backorders will convert into lost sales. In this Appendix we show how this affects the values of t_1, t_2, t_3, t_4, $\bar{B}$ and $\bar{I}$ and, because of this, changes the total cost function $\Gamma(T, F)$ to be minimized. We will then show how this change in the cost function changes the equations for T^* and F^* and the condition on β that must be satisfied for partial backordering to be optimal. In order to eliminate any possible confusion about the meaning of F, we redefine it as the fraction of the cycle during which there is inventory. This means that, as shown in Fig. 4, $FT = t_3 + t_4$.

C.1. The Values of t_1, t_2, t_3, t_4, $\bar{B}$ and $\bar{I}$

Referring to Figs. 2 and 4 and the reasoning in Appendix A, it is obvious that the equations for t_3, t_4, and $\bar{I}$ are exactly the same for LIFO and for FIFO:

$$t_3 = \frac{FTD}{P}, \quad t_4 = FT\left(1 - \frac{D}{P}\right),$$
$$\bar{I} = \frac{DTF^2}{2}\left(1 - \frac{D}{P}\right). \tag{C1}$$

Using the same reasoning, B is still given by $B = \beta D t_1$ but, due to the LIFO policy, $t_2 = B/(P - D) = \beta D t_1/(P - D)$. Since $t_1 + t_2 = (1 - F)T$, we have

$$t_1 = \frac{(1 - F)T(P - D)}{P - D(1 - \beta)},$$
$$t_2 = \frac{\beta(1 - F)TD}{P - D(1 - \beta)},$$
$$\bar{B} = \frac{\beta(1 - F)^2 TD(P - D)}{2(P - D(1 - \beta))}. \tag{C2}$$

C.2. The revised cost function

Recognizing that lost sales can only occur during Interval 1 of the cycle, the cost function is

$$\Gamma(T, F) = \frac{C_o}{T} + C_h\bar{I} + C_b\bar{B} + C_l D(1 - \beta)\frac{t_1}{T}. \tag{C3}$$

Replacing t_1, $\bar{I}$, and $\bar{B}$ by their expressions from (C1) and (C2), we get

$$\Gamma(T, F) = \frac{C_o}{T} + \frac{C_h DTF^2}{2}\left(1 - \frac{D}{P}\right) + \frac{\beta C_b DT(1 - F)^2(P - D)}{2(P - D(1 - \beta))} + \frac{C_l D(1 - F)(1 - \beta)(P - D)}{P - D(1 - \beta)}. \tag{C4}$$

Taking the partial derivatives of $\Gamma(T, F)$ with respect to T and F, setting them equal to 0, solving for T as a function of F and F as a function of T, and then replacing F in $T(F)$ by the expression for $F(T)$, as was done in the development of the FIFO model in the paper, we get

$$T^* = \sqrt{\frac{2C_o^*}{DC_h^*}\left[\frac{C_h^* + \beta C_b}{\beta C_b}\right] - \frac{[(1 - \beta)C_l]^2}{\beta C_h^* C_b}},$$
$$F^* = \frac{(1 - \beta)C_l + \beta C_b T^*}{T^*(C_h^* + \beta C_b)}, \tag{C5}$$

where $C_o^* = C_o(1 - D(1 - \beta)/P)$ and $C_h^* = C_h(1 - D(1 - \beta)/P)$. With the change from C_o, C_h', and C_b' in the FIFO model to C_o^*, C_h^*, and C_b here, these are exactly the same as the expressions in Eqs. (19) and (18).

Again following the same reasoning as in the development of the FIFO model, we can develop a condition on β similar to that in inequality (21) for the FIFO model. For optimal backordering to be optimal when a LIFO policy is used for filling backorders, β must satisfy the following condition, which is equivalent to the one in Mak [26]:

$$\beta \geqslant \beta^* = 1 - \sqrt{\frac{2C_o^* C_h^*}{DC_l^2}}. \tag{C6}$$

While (C6) can be used to test whether any specific value of β is large enough for the equations in (C5) to give an optimal solution, using it to determine the value of β^* requires a search process since β appears in the formula on the right side of the inequality as part of C_o^* and C_h^*. After going through some algebra, we can transform (C6) into an inequality that can be used directly to determine the value of β^*:

$$\beta^* = 1 - \frac{T_{EPQ}PC_h}{PC_l + T_{EPQ}DC_h} = \frac{PC_l - T_{EPQ}C_h(P-D)}{PC_l + T_{EPQ}DC_h}, \tag{C7}$$

where T_{EPQ} is the optimal cycle time for the basic EPQ model with no backordering.

A proof that the values of T^* and F^* given by (C5) are optimal if a LIFO policy is followed and β is at least as large as β^* given by (C7) follows the same outline as in Appendix B if a FIFO policy is used.

References

[1] Pentico DW, Drake MJ. The deterministic EOQ with partial backordering: a new approach. European Journal of Operational Research 2008; in press.

[2] Harris F. How many parts to make at once. Factory, The Magazine of Management 1913;10:135–6, 152. Reprinted in Operations Research 1990;38:947–50.

[3] Woolsey RED. A requiem for the EOQ: an editorial. Production and Inventory Management Journal 1988;29:68–72.

[4] Jagieta L, Michenzi AR. Inflation's impact on the economic lot quantity (EOQ) formula. Omega 1982;10:698–9.

[5] Khouja M, Park S. Optimal lot sizing under continuous price decrease. Omega 2003;31:539–45.

[6] Zipkin PH. Foundations of inventory management. New York: McGraw-Hill; 2000.

[7] Wee HM, Yu J, Chen MC. Optimal inventory model for items with imperfect quality and shortage backordering. Omega 2007;35:7–11.

[8] Montgomery DC, Bazaraa MS, Keswani AK. Inventory models with a mixture of backorders and lost sales. Naval Research Logistics Quarterly 1973;20:255–63.

[9] Rosenberg D. A new analysis of a lot-size model with partial backordering. Naval Research Logistics Quarterly 1979;26: 349–53.

[10] Park KS. Inventory model with partial backorders. International Journal of Systems Science 1982;13:1313–7.

[11] San José LA, Sicilia J, García-Laguna J. The lot size-reorder level inventory system with customers impatience functions. Computers & Industrial Engineering 2005;49:349–62.

[12] San José LA, Sicilia J, García-Laguna J. An inventory system with partial backlogging modeled according to a linear function. Asia-Pacific Journal of Operations Research 2005;22:189–209.

[13] San José LA, Sicilia J, García-Laguna J. Analysis of an inventory system with exponential partial backordering. International Journal of Production Economics 2006;100: 76–86.

[14] San José LA, Sicilia J, García-Laguna J. An economic lot-size model with partial backlogging hinging on waiting time and shortage period. Applied Mathematical Modeling 2007;31: 2149–59.

[15] Abad PL. Optimal pricing and lot-sizing under conditions of perishability and partial backordering. Management Science 1996;42:1093–104.

[16] Abad PL. Optimal price and order size for a reseller under partial backordering. Computers & Operations Research 2001;28:53–65.

[17] Dye C-Y. Joint pricing and ordering policy for a deteriorating inventory with partial backlogging. Omega 2007;35:184–9.

[18] Wee H-M. Deteriorating inventory model with quantity discount, pricing and partial backordering. International Journal of Production Economics 1999;59:511–8.

[19] Chang H-J, Dye C-Y. An EOQ model for deteriorating items with time varying demand and partial backlogging. Journal of the Operational Research Society 1999;50:1176–82.

[20] Yang PC, Wee H-M, Wee KP. An integrated vendor-buyer model with perfect and monopolistic competitions: an educational note. International Transactions in Operations Research 2006;13:75–83.

[21] Abad PL. Optimal lot size for a perishable good under conditions of finite production and partial backordering and lost sale. Computers & Industrial Engineering 2000;38:457–65.

[22] Goyal SK, Giri BC. The production-inventory problem of a product with time varying demand, production and deterioration rates. European Journal of Operational Research 2003;147: 549–57.

[23] Giri BC, Jalan AK, Chaudhuri KS. An economic production lot size model with increasing demand, shortages and partial backlogging. International Transactions in Operations Research 2005;12:235–45.

[24] Lo S-T, Wee H-M, Huang W-C. An integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation. International Journal of Production Economics 2007;106:248–60.

[25] Jolai F, Tavakkoli-Moghaddam R, Rabbani M, Sadoughian MR. An economic production lot size model with deteriorating items, stock-dependent demand, inflation, and partial backordering. Applied Mathematics and Computation 2006;181:380–9.

[26] Mak ML. Determining optimal production-inventory control policies for an inventory system with partial backlogging. Computers & Operations Research 1987;14:299–304.

[27] Li J, Wang S, Cheng TCE. Analysis of postponement strategy by EPQ-based models with planned backorders. Omega 2008;36:777–88.

[28] Chakraborty T, Giri BC, Chaudhuri KS. Production lot sizing with process deterioration and machine breakdown under inspection schedule. Omega 2007, in press, doi:10.1016/j.omega.2006.12.001.