TOPOLOGICAL COMPONENTS OF THE SET OF COMPOSITION OPERATORS ON $H^\infty(B_N)$

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Abstract. We characterize the component structure of the space of composition operators acting on $H^\infty(B_N)$, both in the operator norm topology and in the topology induced by the essential norm.

1. Introduction

Let Ω be a region in $\mathbb{C}^N$, $N \geq 1$, and let B be a Banach space of analytic functions on Ω. For an analytic map $\phi : \Omega \to \Omega$, the composition operator C_ϕ acts on B by

$$C_\phi f(x) = f(\phi(x))$$

for all $f \in B$ and $x \in \Omega$. The set of bounded composition operators acting on B will be denoted $\mathcal{C}(B)$.

Composition operators have been studied extensively in the one dimensional case where the underlying region is the unit disk $D := \{z : |z| < 1\}$ and the Banach space of analytic functions is the classical Hardy space H^2. In this setting, Earl Berkson proved in [3] that if $\phi : D \to D$ is an analytic map whose radial limit function satisfies $|\phi(\zeta)| = 1$ for $\zeta \in E \subset \partial D$, then for any analytic self map of the disk $\psi \neq \phi$,

$$\|C_\phi - C_\psi\| \geq \sqrt{\frac{\sigma(E)}{2}}$$

where σ denotes normalized Lebesgue measure on the unit circle. In other words, any analytic self map of the disk which assumes radial limits of modulus 1 on a set of positive measure is *isolated* in $\mathcal{C}(H^2)$ in the operator norm topology. Other conditions for both isolation and non-isolation in $\mathcal{C}(H^2)$ were provided by Joel Shapiro and Carl Sundberg in [11], who concluded their paper with a number of problems:

i) Characterize the components of $\mathcal{C}(H^2)$.

ii) Characterize isolated elements of $\mathcal{C}(H^2)$.

iii) Characterize which composition operators have compact difference on H^2.

Subsequent inquiry into the topological structure of sets of composition operators has extended to include other spaces and other topologies; see the references in [4], Chapter 9, Section 2. In [9], Barbara MacCluer, Shûichi Ohno, and Ruhan Zhao consider composition operators acting on $H^\infty(D)$; they provide a geometric

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condition for when two composition operators lie in the same component, characterize compact composition operator difference, and use these results to provide an example of two composition operators with non-compact difference that lie in the same component. Takuya Hosokawa, Keiji Izuchi, and Dechao Zheng continue this investigation in [8], where they show, among other things, that a composition operator that is isolated in the norm topology is also isolated in the essential norm topology.

In this paper we extend the results of [8] and [9] to the setting of several variables. After standardizing notation and collecting a few preliminary results in the next section, we devote Section 3 to characterizing components in the operator norm topology. Section 4 characterizes when two composition operators have compact difference, and Section 5 shows that the components under the essential operator norm coincide with those under the usual operator norm. Section 6 provides a class of examples, extensions to the Bloch space, and suggestions for further study.

After this work was completed, the author learned that a number of the same results were obtained independently in [7] by Pamela Gorkin, Raymond Mortini, and Daniel Suárez.

2. Preliminary Results and Definitions

We begin by standardizing notation. We will frequently write (z_1, z') for an element $(z_1, \ldots, z_N) \in \mathbb{C}^N$; if the last $n-1$ components are 0, this will become $(z_1, 0')$. For $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in $\mathbb{C}^N$, we denote the inner product of z and w by

$$< z, w > := z_1 \overline{w_1} + \cdots + z_N \overline{w_N},$$

and define $|z| :=< z, z >^{1/2}$. The symbol B_N will denote the unit ball of $\mathbb{C}^N$, i.e.

$$B_N := \{z \in \mathbb{C}^N : |z| < 1\},$$

and $H^\infty(B_N)$ (also denoted H_N^∞) will denote the Banach space of bounded analytic functions $f : B_N \to \mathbb{C}$ equiped with the supremum norm, which is given by

$$\|f\|_\infty := \sup_{z \in B_N} |f(z)|.$$

The symbol $S(B_N)$ will denote the set of holomorphic maps from B_N to B_N. Since every $\phi \in S(B_N)$ induces a bounded operator C_ϕ on H_N^∞, we have that

$$\mathcal{C}(H_N^\infty) := \{C_\phi : \phi \in S(B_N)\}.$$

Unless otherwise specified, we will assume $\mathcal{C}(H_N^\infty)$ carries the operator norm topology.

For an element $z \in \mathbb{C}$, let $[z]$ denote the complex subspace spanned by z. The canonical automorphism of B_N that interchanges z and 0 is given by

$$\Phi_z(w) := \frac{z - P_z(w) - S_z Q_z(w)}{1 - < w, z >}$$

where $P_z(w)$ is projection onto $[z]$, $Q_z(w)$ is projection onto $[z]^\perp$, and $S_z = \sqrt{1 - |z|^2}$. Note that $\Phi_z(w)$ is an involution, and that if $0 < \lambda < 1$, then $\Phi_z(\lambda B_N)$ is the set of all points $w \in B_N$ satisfying

$$\frac{|P_z(w) - C_{z\lambda}|^2}{\lambda^2 \rho_{z\lambda}^2} + \frac{|Q_z(w)|^2}{\lambda^2 \rho_{z\lambda}} < 1, \tag{1}$$

where

$$C_{z\lambda} := \frac{(1-\lambda^2)z}{1-\lambda^2|z|^2} \qquad \text{and} \qquad \rho_{z\lambda} := \frac{1-|z|^2}{1-\lambda^2|z|^2}$$

(see [10], pg. 30 for details). This equation defines an ellipsoid with center $C_{z\lambda}$; observe that $\Phi_z(\lambda B_N) \cap [z]^\perp$ is a ball of radius $\lambda\sqrt{\rho_{z\lambda}}$, while $\Phi_z(\lambda B_N) \cap [z]$ is a disk centered at $C_{z\lambda}$ of radius $\lambda\rho_{z\lambda}$. This disk has an alternative description as a pseudo-hyperbolic disk centered at z:

$$\Phi_z(\lambda B_N) \cap [z] = \left\{ w \in [z] : \left| \frac{z-w}{1-<z,w>} \right| < \lambda \right\}. \tag{2}$$

Note that for any $z \in B_N$ and $0 < \lambda < 1$, we have

$$|C_{z\lambda} - z| = \lambda^2 |z| \rho_{z\lambda}; \tag{3}$$

this estimate will be used later on.

For $z, w \in B_N$, the *induced distance* between z and w is defined as

$$d_\infty(z,w) := \sup_{\|f\|_\infty = 1} |f(z) - f(w)|.$$

If we let

$$\beta(z,w) := \sup\{|f(z)| : \|f\|_\infty = 1, f(w) = 0\}$$

then a theorem of Bear [1] gives that

$$d_\infty(z,w) = \frac{2 - 2\sqrt{1-\beta(z,w)^2}}{\beta(z,w)}$$

Lemma 1. *For any z in B_N, we have*
a) $\beta(z,w) = |\Phi_z(w)|$ for any $w \in B_N$, and
b) $\{w : \beta(z,w) < \lambda\} = \Phi_z(\lambda B_N)$.

Proof. Since H_N^∞ is Moebius invariant, so must be β, whence

$$\beta(z,w) = \beta(\Phi_z(z), \Phi_z(w)) = \beta(0, \Phi_z(w)).$$

By the Schwarz lemma, $\beta(0, \Phi_z(w)) = |\Phi_z(w)|$, which proves part (a). Part (b) is an immediate consequence of (a) and the involution property of Φ_z. □

Let $\phi, \psi : B_N \to B_N$, and define

$$d_\beta(\phi,\psi) := \sup_{z \in B_N} \beta(\phi(z), \psi(z)).$$

This defines a $[0,1]-$valued metric on $S(B_N)$. Denote the ensuing topological space by $S(B_N, d_\beta)$; the next lemma, whose one variable variant appears in [9], shows that $S(B_N, d_\beta)$ and $\mathcal{C}(H_N^\infty)$ are homeomorphic.

Lemma 2. *Let $\phi, \psi : B_N \to B_N$. Then* $\|C_\phi - C_\psi\| = \frac{2-2\sqrt{1-d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)}$.

Proof.

$$\begin{aligned}\|C_\phi - C_\psi\| &= \sup_{\|f\|_\infty=1} \sup_{z\in B_N} |f(\phi(z)) - f(\psi(z))| \\ &= \sup_{z\in B_N} \sup_{\|f\|_\infty=1} |f(\phi(z)) - f(\psi(z))| \\ &= \sup_{z\in B_N} d_\infty(\phi(z),\psi(z)) \\ &= \sup_{z\in B_N} \frac{2-2\sqrt{1-\beta(\phi(z),\psi(z))^2}}{\beta(\phi(z),\psi(z))} \\ &= \frac{2-2\sqrt{1-d_\beta(\phi,\psi)^2}}{d_\beta(\phi,\psi)}.\end{aligned}$$

□

The fact that $C(H_N^\infty)$ and $S(B_N, d_\beta)$ are homeomorphic now follows from the observation that the function

$$f(x) := \frac{2-2\sqrt{1-x^2}}{x}$$

maps $(0,1]$ continuously onto $(0,2]$ and increases monotonically in x.

3. Components in $\mathcal{C}(H_N^\infty)$

The following theorem gives a geometric condition for when two composition operators lie in the same component of H_N^∞. When $N = 1$, this result appears in [9].

Theorem 1. *Let ϕ and ψ be analytic self maps of B_N. Then the following are equivalent:*
(i) C_ϕ and C_ψ are in the same component in $\mathcal{C}(H_N^\infty)$.
(ii) $d_\beta(\phi,\psi) < 1$.
(iii) $\|C_\phi - C_\psi\| < 2$.

We postpone the proof until we have established a number of rather technical lemmas that will facilitate the implication $(ii) \Rightarrow (i)$.

We begin by introducing some abusive but convenient notation. Given two points z and w in B_N define

$$z_t := (1-t)z + tw. \tag{4}$$

Analogously, given two self maps of the ball ϕ and ψ, we define

$$\phi_t(z) := (1-t)\phi(z) + t\psi(z). \tag{5}$$

Note that $z_0 = z, z_1 = w$, and z_t lies on the straight line connecting z and w; in particular, any convex set that contains z and w also contains z_t.

Lemma 3. *Let z and w lie in B_N, and let $\lambda \in [0,1]$. If $\beta(z,w) < \lambda$, then $\beta(z,z_t) < \lambda$ and $\beta(w,z_t) < \lambda$ for all $0 \le t \le 1$.*

Proof. The image of λB_N under Φ_z is convex and by Lemma 1(b) it contains both z and w. It therefore contains z_t, whereby another application of Lemma 1(b) shows that $\beta(z,z_t) < \lambda$. A symmetrical argument using Φ_w shows that $\beta(w,z_t) < \lambda$. □

Note that a consequence of Lemma 3 is that $\beta(z, z_t) \leq \beta(z, z_\delta)$ if $t < \delta$; this follows by observing that z_t lies on the line segment $[z, z_\delta]$ and writing $z_t = (1-r)z + rz_\delta$ for $r \in (0, 1)$.

Lemma 4. *Let ϵ and λ be given, and satisfy $0 < \epsilon < \lambda < 1$. Then there exists a $\delta = \delta(\epsilon, \lambda) > 0$ such that if z and w in B_N satisfy $\beta(z, w) < \lambda$, then $\beta(z, z_t) < \epsilon$ and $\beta(w, z_{1-t}) < \lambda$ whenever $t < \delta$.*

Proof. By symmetry, it suffices to prove the result for $\beta(z, z_t)$, and since $\beta(z, z_t) < \beta(z, z_\delta)$ if $t < \delta$, it suffice to show that $\beta(z, z_\delta) < \epsilon$. Set

$$\delta := \frac{\epsilon}{4\lambda}(1-\lambda)^2$$

and suppose $\beta(z, w) < \lambda$, which by (1) and Lemma 1(b) is equivalent to supposing that

$$\frac{|P_z(w) - C_{z\lambda}|^2}{\lambda^2 \rho_{z\lambda}^2} + \frac{|Q_z(w)|^2}{\lambda^2 \rho_{z\lambda}} < 1. \tag{6}$$

We need to show that $\beta(z, z_\delta) < \epsilon$, i.e. that

$$\frac{|P_z(z_\delta) - C_{z\epsilon}|^2}{\epsilon^2 \rho_{z\epsilon}^2} + \frac{|Q_z(z_\delta)|^2}{\epsilon^2 \rho_{z\epsilon}} < 1. \tag{7}$$

This is just a calculation, whose details occupy the rest of the proof.

Write

$$\frac{|P_z(z_\delta) - C_{z\epsilon}|}{\epsilon \rho_{z\epsilon}} \leq \frac{|P_z(z_\delta) - z|}{\epsilon \rho_{z\epsilon}} + \frac{|z - C_{z\epsilon}|}{\epsilon \rho_{z\epsilon}}.$$

By (3),

$$\frac{|z - C_{z\epsilon}|}{\epsilon \rho_{z\epsilon}} = \frac{\epsilon^2 |z| \rho_{z\epsilon}}{\epsilon \rho_{z\epsilon}} < \epsilon < \lambda.$$

Moreover,

$$\begin{aligned} \frac{|P_z(z_\delta) - z|}{\epsilon \rho_{z\epsilon}} &= \frac{\delta |P_z(w) - z|}{\epsilon \rho_{z\epsilon}} \\ &= \delta \cdot \frac{|P_z(w) - z|}{\lambda \rho_{z\lambda}} \cdot \frac{\lambda \rho_{z\lambda}}{\epsilon \rho_{z\epsilon}} \\ &\leq \delta \cdot 2 \cdot \frac{\lambda \rho_{z\lambda}}{\epsilon \rho_{z\epsilon}} \\ &< \frac{1-\lambda}{2} \end{aligned}$$

where the third line follows from the fact that $|P_z(w) - z| \leq 2\lambda \rho_{z\lambda}$ (since, by (6), both z and $P_z(w)$ lie in a circle of radius $\lambda \rho_{z\lambda}$), and the fourth line follows from our choice of δ. We conclude that

$$\frac{|P_z(z_\delta) - C_{z\epsilon}|^2}{\epsilon^2 \rho_{z\epsilon}^2} < \left(\lambda + \frac{1-\lambda}{2}\right)^2 = \left(\frac{1+\lambda}{2}\right)^2. \tag{8}$$

Lastly, write

$$\frac{|Q_z(z_\delta)|^2}{\epsilon^2 \rho_{z\epsilon}} = \frac{\delta^2 |Q_z(w)|^2}{\epsilon^2 \rho_{z\epsilon}} = \delta^2 \cdot \frac{|Q_z(w)|^2}{\lambda^2 \rho_{z\lambda}} \cdot \frac{\lambda^2 \rho_{z\lambda}}{\epsilon^2 \rho_{z\epsilon}} \leq \left(\frac{1-\lambda}{2}\right)^2, \tag{9}$$

where the last inequality follows from the fact that $|Q_z(w)|^2 \leq \lambda^2 \rho_{z\lambda}$ (again by (6)), our choice of δ, and a little calculation. Now (8) and (9) together give (7). This proves the lemma. □

Lemma 5. *Let $0 < \epsilon < \lambda < 1$. Then there exists a $\delta = \delta(\epsilon, \lambda) > 0$ such that whenever z and w satisfy $\beta(z, w) < \lambda$ and $|s - t| < \delta$, then $\beta(z_s, z_t) < \epsilon$.*

Proof. By symmetry, it suffices to show this result for $s < t$ with $s < 1/2$. Let ϵ and λ be fixed, find a δ as in the conclusion of Lemma 4, and suppose $|s-t| < \delta/2$. Since z_t lies on the line segment $[z_s, w]$, we can find a number $r \in (0, 1)$ such that

$$z_t = (1 - r)z_s + rw.$$

A calculation shows that that $t = r + s - rs$, whereby we see that

$$\delta/2 > t - s = r(1 - s),$$

i.e. that

$$r < \frac{\delta}{2(1-s)} < \delta.$$

Now use Lemmas 3 and 4. □

Lemma 6. *Let ϕ and ψ be analytic self maps of B_N satisfying $d_\beta(\phi, \psi) \le \lambda < 1$, and let ϕ_t be as in (5). Then, for $t \in [0, 1]$ and δ such that $t + \delta \in [0, 1]$, we have*

$$\lim_{|\delta| \to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0.$$

Proof. Since the δ of Lemma 5 is independent of z and w, the result follows by taking the supremum over the ball. □

We are now in a position to prove Theorem 1:

Proof of Theorem 1. Most of the proof is done:

$(i) \Rightarrow (ii)$ First note that the expression

$$\rho_\beta(\phi, \psi) := \frac{1}{2} \log \frac{1 + d_\beta(\phi, \psi)}{1 - d_\beta(\phi, \psi)}$$

defines a $[0, \infty]-$valued metric on $S(B_N)$ that increases monotonically with d_β. In particular, $\rho_\beta(\phi, \psi) = \infty$ if and only if $d_\beta(\phi, \psi) = 1$. If C_ϕ and C_ψ are in the same component, then given any $\epsilon > 0$, there exists a finite sequence of composition operators $\{C_{\phi_i}\}_{i=1}^m$ such that $C_{\phi_0} = C_\phi$, $C_{\phi_m} = C_\psi$, and for all i,

$$\|C_{\phi_{i+1}} - C_{\phi_i}\| < \epsilon. \tag{10}$$

If $\epsilon < 2$, Lemma 2 shows that (10) can hold if and only if $d_\beta(\phi_{i+1}, \phi_i) < 1$, which can happen if and only if $\rho_\beta(\phi_{i+1}, \phi_i) < \infty$. By the triangle inequality, $\rho_\beta(\phi, \psi)$ must be finite, by which it follows that $d_\beta(\phi, \psi) < 1$.

$(ii) \Rightarrow (i)$ Given ϕ and ψ in $S(B_N)$ with $d_\beta(\phi, \psi) < 1$, define ϕ_s is as in (5). That the map $s \mapsto \phi_s$ is a continuous path from ϕ to ψ in $S(B_N, d_\beta)$ is the content of Lemma 6; that the map $s \mapsto C_{\phi_s}$ is continous from C_ϕ to C_ψ in $\mathcal{C}(B_N)$ then follows from Lemma 2.

$(ii) \Leftrightarrow (iii)$ This is a restatement of Lemma 2. □

An immediate consequence of Theorem 1 is the following characterization for when a composition operator C_ϕ is its own isolated component in $\mathcal{C}(H_N^\infty)$ in the operator norm topology. Again, for $N = 1$, this appears in [9].

Theorem 2. *Let $\phi \in S(B_N)$. Then the following are equivalent:*
(i) C_ϕ is isolated in $\mathcal{C}(H_N^\infty)$.
(ii) For any $\psi \in S(B_N)$ with $\phi \neq \psi$, $d_\beta(\phi,\psi) = 1$
(iii) For any $\psi \in S(B_N)$ with $\phi \neq \psi$, $\|C_\phi - C_\psi\| = 2$.

Note that another consequence of the proof Theorem 1 is that the path components and the components coincide in $\mathcal{C}(H_N^\infty)$.

4. Compact Difference

In this section we characterize when $C_\phi - C_\psi : H_N^\infty \to H_N^\infty$ is compact. The one dimensional cases of these results appear in [9].

Our characterization will involve the Bloch space. Recall that a function analytic on the unit ball is said to belong to the Bloch space $\mathcal{B}_N$ if

$$\|f\|_0 := \sup_{z \in B_N} Q_f(z) < \infty$$

where $Q_f(z)$ is defined as

$$Q_f(z) := \sup_{|\zeta|=1} \left\{ \frac{|\nabla f(z) \cdot \zeta|}{H(z,\zeta)^{1/2}} \right\}$$

(see [12]). Here, $\nabla f(z) \cdot \zeta$ is the directional derivative of f at z in the direction of ζ and $H(z,\zeta)$ is the Bergman metric on B_N. The Bloch norm of f is then given by $\|f\|_{\mathcal{B}_N} := |f(0)| + \|f\|_0$. It is known ([12], Proposition 4.5) that there is a finite constant M such that for any function analytic on the ball,

$$\|f\|_{\mathcal{B}_N} \leq M \|f\|_\infty. \tag{11}$$

In [14], Kehe Zhu shows that the Bloch space induced distance on the ball is given by

$$\sup_{\|f\|_{\mathcal{B}_N}=1} |f(z) - f(w)| = \left(\frac{N+1}{8} \right) \log \frac{1+\beta(w,z)}{1-\beta(w,z)}, \tag{12}$$

where the expression on the right is just the distance function induced by the Bergman metric. We begin by using this expression to provide a characterization in terms of the Bloch space for when C_ϕ and C_ψ lie in the same component:

Lemma 7. *C_ϕ and C_ψ are in the same component of $C(H_N^\infty)$ if and only if $C_\phi - C_\psi : \mathcal{B}_N \to H_N^\infty$ is bounded.*

Proof. $C_\phi - C_\psi : \mathcal{B}_N \to H_N^\infty$ will be bounded if and only if there exists a positive number M such that

$$\sup_{\|f\|_{\mathcal{B}_N}=1} \|(C_\phi - C_\psi) f\|_\infty \leq M.$$

By (12), this is the case if and only if

$$\sup_{z \in B_N} \left(\frac{N+1}{8} \right) \log \frac{1+\beta(\phi(z),\psi(z))}{1-\beta(\phi(z),\psi(z))} \leq M,$$

which is the case if and only if

$$\sup_{z \in B_N} \beta(\phi(z),\psi(z)) < 1,$$

i.e. if and only if $d_\beta(\phi,\psi) < 1$. The result now follows from Theorem 1. ∎

For our discussion of compactness, we will need the following, whose proof is a minor modification of that of Theorem 3.4 in [4]:

Proposition 1 (Compactness Criterion). *Let $\mathcal{X}$ and $\mathcal{Y}$ be either $\mathcal{B}_N$ or H_N^∞, and for $i = 1, \ldots, m$, let $\phi_i \in S(B_N)$ and $\alpha_i \in \mathbb{C}$. Then the linear combination of composition operators $\sum_{i=0}^m \alpha_i C_{\phi_i}$ is compact from $\mathcal{X}$ to $\mathcal{Y}$ if and only if whenever f_n is bounded in $\mathcal{X}$ and $f_n \to 0$ uniformly on compact subsets of B_N, then $(\sum_{i=0}^m \alpha_i C_{\phi_i}) f_n \to 0$ in $\mathcal{Y}$.*

In what follows, we will use the notational convention that $\lim_{|\phi|\to 1}(\cdot) = 0$ if $\|\phi\|_\infty < 1$.

Lemma 8. *Suppose $\lim_{|\phi(z)|\to 1} \beta(\phi(z), \psi(z)) = \lim_{|\psi(z)|\to 1} \beta(\phi(z), \psi(z)) = 0$. Then $C_\phi - C_\psi : \mathcal{B}_N^\infty \to H_N^\infty$ is compact.*

Proof. Let f_n be an arbitrary sequence of functions with Bloch norm 1, converging almost uniformly to 0. By Proposition 1, it suffices to show that $\|(C_\phi - C_\psi)f_n\|_\infty \to 0$. To this end, let $\epsilon > 0$ be given, and find an r such that either $|\phi(z)| > r$ or $|\psi(z)| > r$ implies $\beta(\phi(z), \psi(z))) < \epsilon$. Then divide the ball into two regions, $A := \{z : |\phi(z)| \geq r\} \cup \{z : |\psi(z)| \geq r\}$ and its complement, $B_N \backslash A$. On A,

$$\begin{aligned} |f_n(\phi(z)) - f_n(\psi(z))| &\leq \sup_{\|f\|_{\mathcal{B}_N}=1} |f(\phi(z)) - f(\psi(z))| \\ &= \left(\frac{N+1}{8}\right) \log \frac{1 + \beta(\phi(z), \psi(z))}{1 - \beta(\phi(z), \psi(z))} \\ &\leq \left(\frac{N+1}{8}\right) \log \frac{1+\epsilon}{1-\epsilon} \end{aligned}$$

where the second line follows from (12) and the third from the fact that $z \in A$. This can be made arbitrarily small by chosing small ϵ. On the other hand, $B_N \backslash A$ is contained in rB_N, which is a compact subset of B_N. Since $f_n \to 0$ uniformly on compact subsets, it follows that if $z \in B_N \backslash A$, then $|f_n(\phi(z)) - f_n(\psi(z))| \to 0$ as $n \to \infty$. Thus $\|(C_\phi - C_\psi)f_n\| \to 0$, and the result follows. □

Lemma 9. *Suppose $C_\phi - C_\psi : \mathcal{B}_N \to H_N^\infty$ is compact. Then so is $C_\phi - C_\psi : H_N^\infty \to H_N^\infty$.*

Proof. This follows from (11) and Proposition 1. □

Lemma 10. *Let ϕ and ψ be holomorphic self maps of the unit ball B_N. If $C_\phi - C_\psi : H_N^\infty \to H_N^\infty$ is compact, then $\lim_{|\phi(z)|\to 1} \beta(\phi(z), \psi(z)) = \lim_{|\psi(z)|\to 1} \beta(\phi(z), \psi(z)) = 0$.*

Proof. Suppose that $C_\phi - C_\psi$ is compact but that there exists a sequence $x_n \in B_N$ and a number $\delta > 0$ such that either $|\phi(x_n)| \to 1$ or $|\psi(x_n)| \to 1$ and $\beta(\phi(x_n), \psi(x_n)) \geq \delta$ for all n. Without loss of generality, suppose $|\phi(x_n)| \to 1$. We will obtain a contradiction to Proposition 1 by finding a bounded sequence of functions h_n converging uniformly on compact subsets of B_N to 0 with $\liminf_{n\to\infty} \|(C_\phi - C_\psi)h_n\|_\infty \geq \delta^2/2$.

For ease of notation, relabel the sequences $\{\phi(x_n)\} := \{z_n\}$ and $\{\psi(x_n)\} := \{w_n\}$. By (1) and Lemma 1(b), the hypothesis that $\beta(z_n, w_n) \geq \delta$ is equivalent to

$$\frac{|P_{z_n}(w_n) - C_{z_n\delta}|^2}{\delta^2 \rho_{z_n\delta}^2} + \frac{|Q_{z_n}(w_n)|^2}{\delta^2 \rho_{z_n\delta}} \geq 1. \tag{13}$$

Define a sequence of functions

$$f_n(z) := \frac{1-|z_n|}{1-<z,z_n>}$$

Note that each f_n is bounded in norm by 1, and the f_n converge uniformly on compact subsets of B_N to zero. By Proposition 1, $\|(C_\phi - C_\psi)f_n\|_\infty$ must converge to 0. On the other hand, a calculation shows that

$$\begin{aligned}\|(C_\phi - C_\psi)f_n\|_\infty &= \sup_{z\in B_N} |f_n(\phi(z)) - f_n(\psi(z))| \\ &\geq |f_n(\phi(x_n)) - f_n(\psi(x_n))| \\ &= (1-|z_n|)\left|\frac{1}{1-<z_n,z_n>} - \frac{1}{1-<z_n,w_n>}\right| \\ &= (1-|z_n|)\left|\frac{<z_n,z_n-w_n>}{(1-<z_n,z_n>)(1-<z_n,w_n>)}\right| \\ &= \frac{|z_n|}{1+|z_n|}\left|\frac{z_n - P_{z_n}(w_n)}{1-<z_n,P_{z_n}(w_n)>}\right|\end{aligned}$$

(We have added a projection in the denominator of the last line for uniformity of notation; of course, it changes nothing.) Since the first factor in the expression on the last line goes to $1/2$, the second factor must converge to 0, i.e.

$$\epsilon_n := \left|\frac{z_n - P_{z_n}(w_n)}{1-<z_n,P_{z_n}(w_n)>}\right| \to 0. \tag{14}$$

Now (14) has the same form as (2), which shows that $P_{z_n}(w_n) \in \Phi_{z_n}(\epsilon_n B_N) \cap [z_n]$. This is a disk which contains z_n and has radius $\epsilon_n \rho_{z_n \epsilon_n}$. In particular, $|P_{z_n}(w_n) - z_n| \leq 2\epsilon_n \rho_{z_n\epsilon_n}$, and this is on the order of $\epsilon_n(1-|z_n|^2)$. From this, we see two things: firstly,

$$\begin{aligned}\limsup_{n\to\infty} \frac{|P_{z_n}(w_n) - C_{z_n\delta}|}{\delta\rho_{z_n\delta}} &\leq \limsup_{n\to\infty}\frac{|P_{z_n}(w_n) - z_n|}{\delta\rho_{z_n\delta}} + \limsup_{n\to\infty}\frac{|z_n - C_{z_n\delta}|}{\delta\rho_{z_n\delta}} \\ &\leq \limsup_{n\to\infty}\frac{2\epsilon_n\rho_{z_n\epsilon_n}}{\delta\rho_{z_n\delta}} + \limsup_{n\to\infty}\frac{\delta^2|z_n|\rho_{z_n\delta}}{\delta\rho_{z_n\delta}} \\ &= \limsup_{n\to\infty}\frac{2\epsilon_n}{\delta}\frac{1-\delta|z_n|}{1-\epsilon_n|z_n|} + \limsup_{n\to\infty}\delta|z_n| \\ &= \delta,\end{aligned}$$

where in the second line, the first term is from our estimate on $|P_{z_n}(w_n) - z_n|$ and the second term uses (3). By (13), it now follows that

$$\liminf_{n\to\infty}\frac{|Q_{z_n}(w_n)|^2}{\delta^2\rho_{z_n\delta}} \geq 1-\delta^2,$$

which, upon writing out the definition of $\rho_{z_n\delta}$, is seen to be equivalent to

$$\liminf_{n\to\infty}\frac{|Q_{z_n}(w_n)|^2}{1-|z_n|^2} \geq \delta^2. \tag{15}$$

The second thing that our estimate on $|P_{z_n}(w_n) - z_n|$ shows is that

$$\lim_{n\to\infty}\frac{1-|z_n|^2}{1-<w_n,z_n>} = 1. \tag{16}$$

This follows by writing $1- < z_n, w_n >= 1 - |z_n|^2 + < z_n, w_n - z_n >$ and observing that $| < z_n, w_n - z_n > | \leq |P_{z_n}(w_n) - z_n| \sim \epsilon_n (1 - |z_n|^2)$.

Now define another sequence of functions

$$g_n(z) := \frac{\left\langle z, \frac{Q_{z_n}(w_n)}{\|Q_{z_n}(w_n)\|} \right\rangle^2}{1- < z, z_n >}.$$

Note that for $z \in B_N$,

$$|g_n(z)| \leq \frac{|Q_{z_n}(z)|^2}{1 - |P_{z_n}(z)|} \leq 2,$$

and that $g_n(z_n) = 0$ while $g_n(w_n) = \frac{|Q_{z_n}(w_n)|^2}{1 - < w_n, z_n >}$.

The sequence g_n does not converge uniformly to 0 on compact subsets of B_N, but it is easy to convert into one that does: define a new sequence of functions

$$h_n := g_n \cdot f_n.$$

Since both g_n and f_n are uniformly bounded, so is h_n, and since f_n goes to 0 uniformly on compact subsets of the ball, h_n does as well. By Proposition 1, $\|(C_\phi - C_\psi)h_n\|$ should converge to zero as $n \to \infty$. However, recalling that $\phi(x_n) = z_n$ and $\psi(x_n) = w_n$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|(C_\phi - C_\psi)h_n\|_\infty &= \limsup_{n\to\infty} \sup_{z \in B_N} |h_n(\phi(z)) - h_n(\psi(z))| \\
&\geq \limsup_{n\to\infty} |h_n(\phi(x_n)) - h_n(\psi(x_n))| \\
&= \limsup_{n\to\infty} \left| 0 - \frac{|Q_{z_n}(w_n)|^2}{1- < w_n, z_n >} \cdot \frac{1 - |z_n|}{1- < z_n, w_n >} \right| \\
&= \limsup_{n\to\infty} \frac{|Q_{z_n}(w_n)|^2}{1 - |z_n|^2} \cdot \frac{1 - |z_n|^2}{|1- < w_n, z_n > |} \cdot \frac{1 - |z_n|}{|1- < w_n, z_n > |} \\
&\geq \delta^2/2,
\end{aligned}$$

where the last line follows from (15) and (16). This proves the lemma. □

Taken together, the last three lemmas give the following:

Theorem 3. *Let ϕ and ψ be analytic self maps of the unit ball. The following are equivalent:*
(i) $C_\phi - C_\psi : H_N^\infty \to H_N^\infty$ is compact
(ii) $C_\phi - C_\psi : \mathcal{B}_N \to H_N^\infty$ is compact
(iii) $\lim_{|\phi(z)|\to 1} \beta(\phi(z), \psi(z)) = \lim_{|\psi(z)|\to 1} \beta(\phi(z), \psi(z)) = 0.$

Proof. $(iii) \Rightarrow (ii)$, $(ii) \Rightarrow (i)$, and $(i) \Rightarrow (iii)$ are Lemmas 8, 9, and 10, respectively. □

5. Essential Isolation

The *essential norm* of an operator T on a Banach space B is defined as

$$\inf_K \|T - K\|, \qquad K \text{ a compact operator.}$$

A composition operator in $\mathcal{C}(H_N^\infty)$ is *essentially isolated* if it is isolated in the essential norm topology. In [8], a technique involving *asymptotically interpolating sequences* is used in the one variable setting to show that if C_ϕ is norm isolated, then it is essentially isolated. Here, we use a different technique, motivated to a large extent by a suggestion of Pamela Gorkin in [6], to extend this to the ball of $\mathbb{C}^N$, $N > 1$. The key is the following theorem of Berndtsson [2]:

Proposition 2. *Let $\{x_i\}$ be a sequence in the ball satisfying*

$$\forall k, \qquad \prod_{j\neq k} |\Phi_{x_j}(x_k)| \geq \delta > 0. \tag{17}$$

Then there exists a number $M = M(\delta) < \infty$ and a sequence of functions $F_k \in H_N^\infty$ such that
(i) $F_k(x_j) = \delta_{kj}$
(ii) $\sum_k |F_k(z)| \leq M$ for $|z| < 1$
(The symbol δ_{kj} is equal to 1 if $i = j$ and 0 otherwise.)

For fixed $\delta \in (0, 1)$, we define M_δ to be the smallest M that satisfies the conclusion of Berndtsson's Theorem for all sequences $\{x_i\}$ satisfying (17). Since a sequence that satisfies (17) for some $\delta > 0$ also satisfies (17) for any number less than δ, it is clear that M_δ decreases monotonically as δ increases.

To apply Berndtsson's theorem, we will need the following lemmas:

Lemma 11. *For z_n and w_n in B_N, let $\{(z_n, w_n)\}$ be a sequence of pairs such that $|z_n| \to 1$, $|w_n| \to 1$, and $|\Phi_{z_n}(w_n)| \to 1$ as $n \to \infty$. Then given any $\delta \in (0, 1)$, there exists a subsequence $\{(z_{n_j}, w_{n_j})\}$ such that for this δ, the sequence*

$$\{x_i\} := \{z_{n_1}, w_{n_1}, z_{n_2}, w_{n_2}, \dots\} \tag{18}$$

satisfies (17).

Proof. We claim that we can choose the n_j in such a way that if $\{x_i\}$ is as in (18), then

$$|\Phi_{x_j}(x_k)| \geq 1 - 2^{-\lfloor \frac{k+1}{2} \rfloor} \qquad \text{for all } j < k. \tag{19}$$

where we use the notation

$$\lfloor t \rfloor := \text{greatest integer less than or equal to } t.$$

Now a sequence that satisfies (19) certainly satisfies (17), with

$$\prod_{j\neq k} |\Phi_{x_j}(x_k)| \geq \prod_{i=1}^{\infty} (1 - 2^{-i})^2 := \delta_0 > 0,$$

and by dropping a finite number of terms from the beginning of this sequence, we can arrange to have this product as close to 1 as we like; in particular, we can make it larger than δ. To find the n_j, proceed by induction: choose n_1 such that $|\Phi_{z_{n_1}}(w_{n_1})| > 1/2$. Then the sequence $\{x_1, x_2\}$ satisfies (19). Assume (19) is satisfied for the first $2m$ terms of (18). For the next two terms, use the fact that both $|z_n|$ and $|w_n|$ go to 1, together with the fact that the finite set $\{x_i\}_{i=1}^{2m}$ is contained in a compact set of B_N, to conclude that for n suitably large

$$|\Phi_{x_i}(z_n)| > 1 - 1/2^{m+1} \qquad \text{and} \qquad |\Phi_{x_i}(w_n)| > 1 - 1/2^{m+1} \qquad \text{for } 1 \leq i \leq 2m.$$

Choose n_{m+1} this large, and also large enough that $|\Phi_{z_{n_{m+1}}}(w_{n_{m+1}})| > 1 - 1/2^{m+1}$. Setting $x_{2m+1} = z_{n_{m+1}}$ and $x_{2m+2} = w_{n_{m+1}}$ completes the induction. □

Using a similar idea, it is easy to show the following:

Lemma 12. *Let $\{z_n\}$ be a sequence with $|z_n| \to 1$ as $n \to \infty$. Then given any $\delta \in (0, 1)$ there is a subsequence such that for this δ, $\{x_i\} := \{z_{n_i}\}$ satisfies (17).*

The following will be critical in getting a lower bound on the essential norm of the difference of two composition operators. The idea was taken from [6].

Lemma 13. *Let F_k be a sequence of H^∞ functions such that $\sum_0^\infty |F_k(z)| \leq M < \infty$ for all $z \in B_N$. Then $F_k \to 0$ weakly.*

Proof. Let $\lambda \in (H_N^\infty)^*$. For any integer N, there exists some unimodular sequence α_n such that

$$\sum_0^N |\lambda F_k| = \lambda \sum_0^N \alpha_n F_k \leq \|\lambda\| \cdot \|\sum_0^N \alpha_n F_k\| \leq \|\lambda\| \cdot M.$$

Thus $\lambda(F_k) \to 0$. □

Recall in what follows that the weak convergence in H_N^∞ implies uniform convergence on compact subsets of B_N.

Theorem 4. *Suppose C_ϕ is norm isolated in $\mathcal{C}(H_N^\infty)$. Then C_ϕ is also essentially isolated.*

Proof. Suppose C_ϕ is isolated, and let $\psi \in S(B_N)$, $\psi \neq \phi$. Fix $\delta \in (0,1)$. We will show that $\|C_\phi - C_\psi\|_e \geq \frac{1}{M_\delta}$.

By Theorem 1, we can find a sequence z_n such that $|\Phi_{\phi(z_n)}(\psi(z_n))| \to 1$ as either $|\phi(z_n)| \to 1$ or $|\psi(z_n)| \to 1$. Without loss of generality, suppose the former. If some subsequence of the $\psi(z_n)$ converges to $a \in B_N$ with $|a| < 1$, use Lemma 12 to pass to another subsequence $\{z_{n_i}\}$ such that this convergence obtains and also $\{x_i\} := \{\phi(z_{n_i})\}$ satisfies (17) for our chosen δ. Find a sequence F_k as in Proposition 2, with $\|F_k\|_\infty \leq M_\delta$. Let K be any compact operator, and calculate:

$$\begin{aligned} \|C_\phi - C_\psi + K\|_\infty &\geq \limsup_{k\to\infty} \frac{1}{M_\delta} \|(C_\phi - C_\psi - K)F_k\| \\ &\geq \limsup_{k\to\infty} \frac{1}{M_\delta} \left(\|(C_\phi - C_\psi)F_k\| - \|KF_k\|\right) \\ &= \limsup_{k\to\infty} \frac{1}{M_\delta} \left(\|(C_\phi - C_\psi)F_k\|\right) \\ &\geq \frac{1}{M_\delta}. \end{aligned} \tag{20}$$

The second to last line follows from the compactness of K and the weak null covergence of the F_k; the last line follows from the fact that $F_k(\phi(z_k)) = 1$, $F_k \to 0$ weakly, and $\{\psi(z_{n_i})\}$ is contained in a compact subset of the ball. We conclude that $\|C_\phi - C_\psi\|_e \geq \frac{1}{M_\delta}$.

In the case where $|\psi(z_n)| \to 1$, use Lemma 11 to pass to a subsequence $\{z_{n_j}\}$ such that for our chosen δ, $\{x_i\} := \{\phi(z_{n_1}), \psi(z_{n_1}), \phi(z_{n_2}), \psi(z_{n_2}), \dots\}$ satisfies (17). Find functions F_k satisfying the conclusion of Proposition 2 with $\|F_k\|_\infty \leq M_\delta$, and calculate as in (20) to show that $\|C_\phi - C_\psi\|_e \geq \frac{1}{M_\delta}$. □

Note that since δ was arbitrary, the proof actually shows that $\|C_\phi - C_\psi\|_e \geq \lim_{\delta\to 1} \frac{1}{M_\delta}$.

Theorem 4 says that the singleton components of $\mathcal{C}(H_N^\infty)$ coincide in the norm and essential norm topologies. In fact, the proof really shows that whenever

$d_\beta(\phi,\psi) = 1$, then $\|C_\phi - C_\psi\|_e \geq \frac{1}{M_\delta}$. This can be used to show that all components are identical under these two topologies:

Theorem 5. *The components in $\mathcal{C}(H^\infty_N)$ coincide in the norm and essential norm topologies.*

Proof. If C_ϕ and C_ψ are in the same component in the norm topology, then they are clearly in the same component in the essential norm topology. Conversely, suppose that C_ϕ and C_ψ are in different components in the norm topology. By Theorem 1, $d_\beta(\phi,\psi) = 1$. If C_ϕ and C_ψ lay in the same component in the essential norm topology, then given any $\epsilon > 0$, we could find a finite chain C_{ϕ_i}, $i = 1,\dots,m$, satisfying $C_{\phi_0} = C_\phi$, $C_{\phi_m} = C_\psi$, and $\|C_{\phi_{i+1}} - C_{\phi_i}\|_e < \epsilon$. But since $d_\beta(\phi,\psi) = 1$, the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 1 shows that $d_\beta(\phi_{i+1},\phi_i) = 1$ for at least one i, and then the proof of Theorem 4 shows that for this i, $\|C_{\phi_{i+1}} - C_{\psi_i}\|_e \geq \frac{1}{M_\delta}$. Choosing $\epsilon < \frac{1}{M_\delta}$ leads to a contradiction. □

6. Concluding Remarks

For the remainder of the paper we return to the norm topology. We first note that any function in one variable can be extended to a function in several variables as follows. For $\psi : D \to D$, define $\Psi : B_N \to B_N$ by

$$\Psi(z) := (\psi(z_1), 0')$$

Then Ψ is analytic if ψ is, and for any two such maps ψ_1 and ψ_2, it is easy to see that

$$d_\beta(\psi_1,\psi_2) = d_\beta(\Psi_1,\Psi_2)$$

(where the d_β on the left is, of course, just the one variable version.) It follows from this equality and Theorem 1 that C_{Ψ_1} and C_{Ψ_2} are in the same component in $\mathcal{C}(H^\infty_N)$ if and only if C_{ψ_1} and C_{ψ_2} are in the same component in $\mathcal{C}(H^\infty_1)$. Similarly, Ψ_1 and Ψ_2 satisfy the compact difference condition of Theorem 3(iii) if and only if ψ_1 and ψ_2 do, i.e. $C_{\Psi_1} - C_{\Psi_2}$ is compact on H^∞_N if and only if $C_{\psi_1} - C_{\psi_2}$ is compact on H^∞_1. In particular, the examples of [8] and [9] extend to show the following:

Corollary 1. *For each $N \geq 1$, there exist composition operators with non-compact difference that lie in the same component of $\mathcal{C}(H^\infty_N)$.*

Next, an example of an isolated composition operator can be constructed by observing that if $\phi,\psi \in S(B_N)$ have different radial limits at a point $\zeta \in \partial B_N$, then $d_\beta(\phi,\psi) = 1$. Since any two distinct maps ϕ and ψ in $S(B_N)$ have different radial limits $\sigma-$almost everywhere (where σ denotes rotation invariant Lebesgue measure on the unit sphere), it follows that whenever ϕ has radial limits of modulus 1 on a set E with $\sigma(E) > 0$, then C_ϕ is isolated.

When $N = 1$, Hosokawa, Izuchi, and Zheng show in [8] that the isolated composition operators are exactly those induced by non-extreme points, i.e. by C_ϕ with

$$\int_{\partial D} \log(1 - |\phi(\zeta)|)d\zeta > -\infty.$$

It would be nice to generalize this to arbitrary N.

Finally, it is easy to use the preceding techniques and results to show that the component structure of $\mathcal{C}(\mathcal{B}_N)$ is in some sense subordinate to the component structure of $\mathcal{C}(H^\infty_N)$. This is the content of the following:

Lemma 14. *If C_ϕ and C_ψ are in the same component in $\mathcal{C}(H_N^\infty)$, then they are in the same component in $\mathcal{C}(\mathcal{B}_N)$.*

Proof. If $d_\beta(\phi,\psi) < 1$, then the proof of Lemma 7 gives that $C_\phi - C_\psi : \mathcal{B}_N \to H_N^\infty$ is bounded with

$$\|C_\phi - C_\psi\|_{\mathcal{B}_N \to H_N^\infty} = \left(\frac{N+1}{8}\right) \log\left(\frac{1+d_\beta(\phi,\psi)}{1-d_\beta(\phi,\psi)}\right).$$

Define $\phi_s(z)$ as in (5), and invoke Lemma 6 to conclude that

$$\lim_{|\delta|\to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0,$$

whence $\|C_{\phi_t} - C_{\phi_{t+\delta}}\|_{\mathcal{B}_N \to H_N^\infty} \to 0$ as $\delta \to 0$. Now use (11) to conclude

$$\begin{aligned} \|C_{\phi_t} - C_{\phi_{t+\delta}}\|_{\mathcal{B}_N} &= \sup_{\|f\|_{\mathcal{B}_N}} \|(C_{\phi_t} - C_{\phi_{t+\delta}})f\|_{\mathcal{B}_N} \\ &\le \sup_{\|f\|_{\mathcal{B}_N}} M\|(C_{\phi_t} - C_{\phi_{t+\delta}})f\|_\infty \\ &= M\|C_{\phi_t} - C_{\phi_{t+\delta}}\|_{\mathcal{B}_N \to H_N^\infty}. \end{aligned}$$

Since this goes to 0 as $\delta \to 0$, the result follows. □

The converse is not true, since it can be shown, for example, that the compact composition operators are connected in $\mathcal{C}(\mathcal{B}_N)$, yet there exist compact composition operators C_ϕ and C_ψ such that $d_\beta(\phi,\psi) = 1$ (see [13] for details).

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