Using the EPQ with Partial Backordering for Coordinated Planning of a Product and its Components

Abstract

While there has been considerable work over the years on multistage lot-sizing models, particularly in an MRP environment, there has been relatively little work on systems recognizing the WIP effects when there is gradual conversion of the components into the final product, as in production planning using the EPQ model for planning the final product. Here we consider lot-sizing planning for a two-stage system in which the final product is planned using an EPQ model with partial backordering and the production of the components is controlled using basic EPQ models without backordering.

Keywords: EPQ, partial backordering, lot-sizing, component planning

1. Introduction

While the basic deterministic economic order quantity (EOQ) and its derivative models, such as the economic production quantity (EPQ) model and the EOQ and EPQ models with either full or partial backordering of demand during stockout periods, have been used for many years, there have been few extensions of these models that recognize the potential effects of work-in-process (WIP) inventories of the components of the final product being planned by the EOQ or EPQ model.

There has been considerable work on developing models for lot-sizing that recognize the effects of WIP inventories in multi-stage production systems. Most of this work, however, has been in the context of batch production systems, primarily based on planning with Material Requirements Planning (MRP) systems. There has been little work on models that recognize, in the words of Banerjee et al. [1], "the gradual conversion of input to output at a finite rate within each production stage," as in systems where EPQ models might be used to plan production. What work there has been is restricted to looking at two-level systems consisting of a single final product and its immediate components.

The first such work was by Banerjee and Burton [2] and Banerjee et al. [1] that assumed that the final product is planned with an EPQ model with no backordering and the components are ordered using basic EOQ models. Sharma and Singh [3] extended this model to include using the EPQ model to plan component production and allowed for less-than-perfect production of the final product.

In this paper we extend this previous work by using Pentico et al.'s [4] model for the single-product EPQ with partial backordering model as the starting point for a model to plan for the coordinated planning of a two-level system consisting of a final product, which has the characteristics of an EPQ with partial backordering, and its immediate components, which are planned with an EPQ model without backordering. In addition, we present three variations on that model that have other possible desirable or useful characteristics. We also examine the results of a numerical study designed to evaluate the performance of the base model and its variations under a variety of conditions that represent combinations of levels for situational characteristics that might be important in determining how well the models perform.

2. Literature review

The production planning literature discusses two basic methods of managing inventory levels of production components and sub-components. The first, MRP, is a push-based method where components are produced or ordered to fulfill an exogenously-developed production schedule for the final product. The second is a pull-based method often operationalized through the use of Kanban production quantities that are authorized when demand for the component is realized downstream in the manufacturing process.

While the original studies only considered single-stage production systems, subsequent work has expanded the scope to multi-stage production systems, some of which also consider the inventory decisions for the final product. Franca et al. [5], for example, present a heuristic for determining MRP lot-sizing rules that minimize cost in a multi-stage, resource-constrained manufacturing system. Bard and Golany [6], on the other hand, determine Kanban requirements in a multi-stage system with multiple products. Several studies, including Rees et al. [7] have compared the two methods in a multi-stage environment, and other studies have expanded the basic multi-stage models to include facets such as stochastic demand for the final product (Grubbstrom and Wang [8]). Another stream of literature (see, e.g., Clark and Scarf [9], Maxwell and Muckstadt [10], and Roundy [11]) considers inventory decisions and replenishments of finished goods in multi-stage distribution networks. Zipkin [12] provides an overview of these so-called "multi-echelon" distribution systems.

Our model differs from these studies and those mentioned in Section 1 in the following important ways. Most prior research considers batch-type production, ignoring the fact that finished goods or components may become available one-at-a-time to satisfy downstream demand while the rest of the units are still in production. Our model also includes the partial backordering phenomenon where only some customers who are faced with a stockout situation will be willing to wait to receive the good. We are aware of only two papers (Mak [13] and Pentico et al. [4]) that address finished-good inventory control decisions in a production environment under partial backordering, and we have not found any prior research that considers multi-stage inventory decisions for finished goods and components in such an environment.

3. The basic model

We extend the models of Banerjee et al. [1] and Sharma and Singh [3] for the coordinated scheduling of the production of a single product and its component parts to include partial backordering of the final product. Our starting point is the model by Pentico et al. [4] for the single-product economic production quantity (EPQ) with partial backordering. After covering the assumptions and notation to be used in our model, we show how that model can be augmented to include ordering and holding inventory of the components.

3.1 Assumptions and notation

3.1.1 Assumptions

For the final product, we make all the usual assumptions of a deterministic EPQ model with full backordering except the full backordering assumption. Instead, as in most other research on partial backordering for either the deterministic EOQ or EPQ, we assume that a constant fraction β of the demands that occur during the stockout period will be backordered. In addition, we assume:

- FIFO backorder filling for the final product, as discussed in Pentico et al. [4], meaning that the oldest backorders are filled first;
- a one-stage final production system, as in Banerjee et al. [1], and, unlike Sharma and Singh [3], we assume that final production has perfect quality; our model can be easily adapted for yield losses in the final production phase, as shown in Appendix A;
- the final product has m components (without loss of generality, we assume that there is one unit of each component in the final product as in Banerjee et al. [1]; if there are two or more units of a component, redefine a unit of component to consist of two, three, etc. original units, adjusting the holding cost and production rate for the component to compensate for the redefinition);
- perfect quality of the components;
- the production of components is according to an EPQ process with no backordering.

3.1.2 Notation

We use the same notation as in Pentico et al. [4], adding terms as needed to include the planning for the components.

Parameters

D = demand per year for the final product

P = production rate per year of the final product

P_i = production rate per year of component i (Note: $P_i > P$ for feasibility of the model)

s = the unit selling price for the final product

C_o = the fixed cost of placing and receiving an order for the final product

C_{oi} = the fixed cost of placing and receiving an order for component i

C_p = the variable cost of producing a unit of the final product

C_{pi} = the variable cost of producing a unit of component i (included in the final product's cost C_p)

C_h = the cost per year to hold a unit of the final product in inventory

C_{hi} = the cost per year to hold a unit of component i in inventory

C_b = the cost per year to keep a unit of the final product backordered

C_g = the goodwill loss on a unit of unfilled demand for the final product

$C_l = (s - C_p) + C_g$ = the cost of a lost sale for the final product, including the lost profit on that unit and any goodwill loss

β = the fraction of stockouts of the final product that will be backordered

Variables

Q = the order quantity for the final product

Q_i = the order quantity for component i

T = the length of an order cycle for the final product

I = the maximum inventory level of the final product, with $\bar{I}$ being the average inventory level over the year

$\bar{I}_i$ = the average inventory level over the year for component i

S = the maximum stockout level of the final product, including both backorders and lost sales

B = the maximum backorder position for the final product, with $\bar{B}$ being the average backorder level over the year ($B = \beta S$)

F = the fill rate or the percentage of demand for the final product that will be filled from stock

N_i = the number of times component i is produced/ordered during a production interval for the final product

3.2 The objective function

We first develop the part of the objective function considering only the final product, and then develop what needs to be added to include the costs associated with the components.

3.2.1 The Costs due to the Final Product

As shown in Pentico et al. [4], the average profit per year, considering only the final product, is the revenue from filling demands, either from stock or as backorders, minus the cost of placing orders, the cost of the units sold, the cost of carrying inventory, the cost of the backorders, and the cost of lost sales. Thus:

$$\Pi(T,F) = (s - C_p)D\,[F + \beta(1-F)] - [C_o/T + C_h\bar{I} + C_b\bar{B} + C_gD(1-\beta)(1-F)]$$
$$= (s - C_p)D - \Gamma(T,F) \tag{1}$$

where

$$\Gamma(T,F) = C_o/T + C_h\bar{I} + C_b\bar{B} + C_lD(1-\beta)(1-F). \tag{2}$$

Since $(s - C_p)D$ is a constant, $\Pi(T,F)$ is maximized by the pair (T,F) that minimizes $\Gamma(T,F)$.

As shown in Pentico et al. [4], the values of $\bar{I}$ and $\bar{B}$ are:

$$\bar{I} = \frac{DTF^2}{2}\left(1 - \frac{D}{P}\right) \quad \text{and} \quad \bar{B} = \frac{\beta DT(1\text{-}F)^2}{2}\left(1 - \frac{\beta D}{P}\right). \tag{3}$$

Substituting these expressions into (2) gives:

$$\Gamma(T,F) = \frac{C_o}{T} + \frac{C_hDTF^2}{2}\left(1 - \frac{D}{P}\right) + \frac{\beta C_bDT(1\text{-}F)^2}{2}\left(1 - \frac{\beta D}{P}\right) + C_lD(1-\beta)(1-F). \tag{4}$$

To simplify the notation, we define $C_h' = C_h(1 - D/P)$ and $C_b' = C_b(1 - \beta D/P)$, which makes the average cost per year, considering only the final product:

$$\Gamma(T,F) = \frac{C_o}{T} + \frac{C_h' DTF^2}{2} + \frac{\beta C_b' DT(1\text{-}F)^2}{2} + C_l D(1-\beta)(1-F). \tag{5}$$

3.2.2 The Costs Due to the Components

Referring to Figure 1, the total time during a production cycle for the final product during which there is inventory of component i is $t_2 + t_3$. From Appendix A of Pentico et al. [4],

$$t_2 + t_3 = (1-F)T(\beta D/P) + FTD/P = \frac{DT}{P}[\beta + (1-\beta)F]. \tag{6}$$

Thus, the use of component i during one production/ordering cycle during a production cycle for the final product is:

$$Q_i = P(t_2 + t_3)/N_i = \frac{PDT}{PN_i}[\beta + (1-\beta)F] = \frac{DT}{N_i}[\beta + (1-\beta)F] \tag{7}$$

and the average inventory of component i is:

$$\bar{I}_i = \frac{Q_i}{2}(1 - \frac{P}{P_i})\left(\frac{t_2 + t_3}{T}\right) = \frac{TD^2[\beta + (1-\beta)F]^2(1 - P/P_i)}{2PN_i}. \tag{8}$$

Since component i is ordered N_i times during each production cycle for the final product, the average annual cost due to ordering and carrying inventory of the components is:

$$\mathrm{K}(N_1, \ldots, N_m | T,F) = \frac{\sum_{i=1}^{m} N_i C_{oi}}{T} + \sum_{i=1}^{m} C_{hi} \bar{I}_i \tag{9}$$

3.2.3 The Complete Objective Function

Adding the average annual costs due to the final product and the average annual costs due to the m components, the overall average annual cost associated with the product and its components is:

$$\Gamma(T,F,N_1,\ldots,N_m) = \frac{C_o + \sum_{i=1}^{m} N_i C_{oi}}{T} + C_h^{'} \frac{DTF^2}{2} + \frac{TD^2[\beta+(1-\beta)F]^2}{2P}\sum_{i=1}^{m}\frac{C_{hi}^{'}}{N_i} + C_b^{'}\frac{\beta DT(1\text{-}F)^2}{2}$$
$$+ C_l D(1-\beta)(1-F) \qquad (10)$$

where, to simplify the notation, $C_{hi}^{'} = C_{hi}(1 - P/P_i)$.

(Note: This model assumes that the manufacturer of the final product also produces the components in a steady stream, so the EPQ assumptions are used. If the components are produced in batches, with all units becoming available at the same time, or are bought from a supplier, then the basic EOQ assumptions are more appropriate. The equations in Sections 4 and 5 can be adapted for this case by replacing $C_{hi}^{'}$ by C_{hi}.)

3.3 Formal problem statement

Given the definitions of T, F, and the N_is above, and recognizing, as we discuss later, that the N_is must be integers to make sense, we can formally state the problem to be solved as:

Minimize $\Gamma(T,F,N_1,\ldots,N_m)$ (Eq. (10))

Subject to:

$T > 0$

$0 \le F \le 1$

$N_1, \ldots, N_m$ positive integers

4. Analytic model

In this section we develop the theoretical background for solving the problem stated at the end of Section 3, describe the process for implementing that solution procedure, and give an example.

4.1 Equations and theoretical results for continuous optimization

While the cost function given in (10) should be optimized over $T \varepsilon (0, \infty)$, $F \varepsilon [0, 1]$, and N_i positive integers, this is a complex, mixed-integer task that is computationally intensive and algorithmically tangled. In this section, therefore, we consider a relaxed version of this problem wherein we consider Γ as a function whose arguments can take any real value. The advantage of the relaxed problem is that it admits solution via the mechanisms of vector calculus, i.e., by finding points where the gradient is zero. The main result of this section is that this relaxed problem has a simple, closed-form analytic solution. In Section 4.3 we will discuss how close this solution is to the integerized one.

In the remainder of this section, the symbol Γ will always represent the relaxed cost function.

Lemma 1. For fixed T and F, there is a unique set of N_i that minimizes Γ.

Proof. Taking the partials of Γ with respect to N_i for each i and setting the results to zero yields a unique solution

$$N_i^* = \frac{DT[\beta + (1-\beta)F]}{\sqrt{2P}} \sqrt{\frac{C'_{hi}}{C_{oi}}} . \tag{11}$$

By vector calculus, N_i^* is the only candidate for an interior minimizer, and since such a minimizer clearly exists (note that the cost goes to infinity as $N_i \to 0$ or $N_i \to \infty$, N_i^* must be that minimizer. □

For fixed T and F, let $N_1^*, \ldots, N_m^*$ be as in (11), and define the function Γ* as

$\Gamma^*(T,F) = \Gamma(T,F, N_1^*, \ldots, N_m^*)$.

That Γ* has a convenient and compact representation is the subject of the following lemma.

Lemma 2. The function Γ* has the form

$$\Gamma^*(T,F) = \frac{G_0}{T} + TR(F) + Q(F) \tag{12}$$

where $R(F)$ and $Q(F)$ are quadratic and linear polynomials, respectively, given by

$$R(F) := G_1F^2 - 2G_2F + G_2$$

$$Q(F) := G_3F + G_4$$

with the coefficients G_i given by

$$G_0 = C_o \qquad G_1 = \frac{D(C_h^{'} + \beta C_b^{'})}{2} \qquad G_2 = \frac{\beta C_b^{'} D}{2} \tag{13}$$

$$G_3 = \frac{2D(1-\beta)}{\sqrt{2P}} \sum_1^m \sqrt{C_{hi}^{'} C_{oi}} - C_l D(1-\beta) \qquad G_4 = \frac{2D\beta}{\sqrt{2P}} \sum_1^m \sqrt{C_{hi}^{'} C_{oi}} - C_l D(1-\beta)$$

Proof. The result follows by simple algebra; substitute (11) into (10) and simplify. □

An advantage of (12) is that it facilitates analysis of the behavior of Γ under changes in T and F. In particular, we can show that Γ has a global optimum.

Lemma 3. The cost function Γ has a unique minimizer.

Proof. Since for each pair (T,F), the N_i^* are unique, it suffices to show that Γ* has a unique minimizer. But, by taking the partial of Γ* with respect to T, we see that for each F there is a unique minimizing T given by

$$T^* = T^*(F) = \sqrt{\frac{G_0}{R(F)}}\,. \tag{14}$$

Substituting this result into Γ* yields a function of the single variable F:

$$\Gamma_T^*(F) := \Gamma^*(T^*,F) = 2\sqrt{G_0 R(F)} + Q(F).$$

The first and second derivatives of Γ_T^* are given by

$$\Gamma_T^{*}{}'(F) = \sqrt{G_0}\,\frac{R'(F)}{(R(F))^{1/2}} + Q'(F) \tag{15}$$

and

$$\Gamma_T^{*}{}''(F) = \frac{\sqrt{G_0}[2R''(F)R(F) - (R'(F))^2]}{2(R(F))^{3/2}} \tag{16}$$

respectively. Since the second derivative factors as

$$\Gamma_T^{*}{}''(F) = \frac{2G_2\sqrt{G_0}}{(R(F))^{3/2}}(G_1 - G_2),$$

which is positive for all F, it follows that $\Gamma_T^{*}(F)$ has at most one minimizer. Thus, so does Γ. □

This proof does not specify where the minimizer lies. In particular, it could lie on the boundary. Here the situation differs somewhat from the result in Pentico et al. [4], where the minimizer was guaranteed to lie in the semi-open interval (0, 1], and whether or not it lay on the boundary or the interior was determined by the value of β. In the present case, it is not just β that determines the position of the minimizer, but also the values of the Gs, which is to say the values of the C_{hi}s and the C_{oi}s. This is the content of the following:

Corollary 1. The value of F that minimizes Γ is 0 if and only if

$$G_3 - 2\sqrt{G_0 G_2} > 0. \tag{17}$$

Proof. The result follows by substituting $F = 0$ into (15) and observing that the result is positive if and only if (17) is met. Since the function Γ_T^{*} is concave up, this means that the cost increases as F does, and thus the minimizer is on the left-hand boundary. □

Corollary 2. If (17) does not hold, then the minimizer lies on the boundary $F = 1$ if and only if

$$G_3 + 2\sqrt{G_0(G_1 - G_2)} < 0. \tag{18}$$

Otherwise, the minimizer is an interior point.

Proof. The result follows as above by substituting $F = 1$ into (15) and observing that the result is negative if and only if (18) holds. □

The above conditions relate to the position of the minimizer. The question remains of how to find it. Another advantage of the form (12) for the cost function is that it allows us to write down an explicit solution. This is the content of the following:

Lemma 4. Let (T^*, F^*) denote the (unique) pair that minimizes (12). Then F^* is the root of a quadratic polynomial

$$P(F) = aF^2 + bF + c \tag{19}$$

with coefficients

$$a = G_1G_3^2 - 4G_0G_1^2 \qquad b = 8G_0G_1G_2 - 2G_2G_3^2 \qquad c = G_2G_3^2 - 4G_0G_2^2 \tag{20}$$

and T^* satisfies

$$T^* = \frac{-G_3}{2(G_1F - G_2)}. \tag{21}$$

F^* should be chosen as the root of (19) that makes T^* positive.

Proof. Generate a system of two equations in the unknowns F and T by taking the partials of Γ^* and equating each to zero. The formulas in the lemma follow by simple algebra. The same algebra shows that the two roots of the polynomial $P(F)$ will yield $\underline{+}|T^*|$ when substituted into (21), and since T is constrained to be positive, the choice of root is clear. □

Note that the solution is formal in the sense there is no guarantee that the pair (T^*, F^*) lies in the domain of Γ^*. Lemma 4 must be used in conjunction with Corollaries 1 and 2 to figure out if the minimizer lies on a boundary. The $F^* = 0$ case is, of course, degenerate; for the $F^* = 1$ case, the corresponding T^* can be found from (14), which yields the basic formula for T^* for the EPQ with no backordering.

4.2 Solution procedure for continuous optimization

Using the results in the lemmas and corollaries in Section 4.1, we define the solution procedure as follows:

Step 1: Use the equations in (13) to find the values of G_0, G_1, G_2, and G_3.

Step 2: From Corollary 1, if $G_3 - 2\sqrt{G_0 G_2} > 0$, set $T = \infty$ and $F = 0$. The product should not be produced. If not, go to Step 3.

Step 3: From Corollary 2, if $G_3 + 2\sqrt{G_0(G_1 - G_2)} < 0$, set $F = 1$ and $T = \sqrt{2C_o/(DC_h^{'})}$. The product should be produced with no backordering. Go to Step 7. If not, go to Step 4.

Step 4: Use the equations in (20) to compute the values of a, b, and c as:

$$a = G_1 G_3^2 - 4G_0 G_1^2 \qquad b = 8G_0 G_1 G_2 - 2G_2 G_3^2 \qquad c = G_2 G_3^2 - 4G_0 G_2^2$$

Step 5: Use the quadratic formula to determine two values of F:

$$F_1, F_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 6: Use F_1 and F_2 to determine two values of T from:

$$T = \frac{-G_3}{2(G_1 F - G_2)}$$

Set $T^* = T_1$ or T_2, whichever is positive. Set F^* = the value of F that gave T^*.

Step 7: Use (11) m times to determine the values of $N_1, N_2, \ldots, N_m$.

We illustrate this procedure with an example, which we will use throughout this paper.

Example – Part 1: Finding the Continuous Optimum Solution

The product has four components. The parameter values are:

Final product: $C_o = 475$, $C_h = 95$, $C_b = 118.75$, $C_l = 296.875$, $D = 10$, $P = 120$, $\beta = 0.75$

Component 1: $C_{o1} = 0.25$, $C_{h1} = 20$, $P_1 = 480$

Component 2: $C_{o2} = 2.0$, $C_{h2} = 18$, $P_2 = 300$

Component 3: $C_{o3} = 0.15$, $C_{h3} = 17$, $P_3 = 180$

Component 4: $C_{o4} = 0.20$, $C_{h4} = 21$, $P_4 = 600$

Using the definitions given in Section 3, compute the values of $C_h^{'}$, $C_b^{'}$, and $C_{hi}^{'}$:

$$C_h^{'} = C_h(1 - D/P),\ C_b^{'} = C_b(1 - \beta D/P),\ C_{hi}^{'} = C_{hi}(1 - P/P_i)$$

$$C_h^{'} = 95(1 - 10/120) = 87.0833 \quad C_b^{'} = 118.75(1 - (0.75)(10)/120) = 111.3281$$

$$C_{h1}^{'} = 20(1 - 120/480) = 15 \qquad C_{h2}^{'} = 18(1 - 120/300) = 10.8$$

$$C_{h3}^{'} = 17(1 - 120/180) = 5.667 \qquad C_{h4}^{'} = 21(1 - 120/600) = 16.8$$

Step 1: Compute the values of G_0, G_1, G_2, and G_3.

$$G_0 = 475 \quad G_1 = 10(87.0833 + (0.75)(111.3281))/2 = 852.90$$

$$G_2 = (0.75)(111.3281)/2 = 417.4805$$

$$G_3 = \frac{2(10)(0.25)}{\sqrt{2(120)}}(9.3391) - (296.875)(10)(0.75) = -739.1733$$

$$G_4 = \frac{2(10)(0.75)}{\sqrt{2(120)}}(9.3391) + (296.875)(10)(0.25) = 751.23$$

Step 2: Since $G_3 - 2\sqrt{G_0 G_2} = -739.1733 - 890.625 = -1629.8$ is not > 0, go to Step 3.

Step 3: Since $G_3 + 2\sqrt{G_0(G_1 - G_2)} = -739.1733 + 909.5557 = 170.4$ is not < 0, go to Step 4.

Step 4: Compute the values of a, b, and c:

$$a = (852.90)(-739.1733)^2 - 4(475)(852.90)^2 = -916{,}120{,}132.9$$

$$b = 8(475)(852.90)(417.4805) - 2(417.4805)(-739.1733)^2 = 896{,}854{,}372.3$$

$$c = (417.4805)(-739.1733)^2 - 4(475)(417.4805)^2 = -103{,}049{,}073.4$$

Step 5: Use the quadratic formula to determine two values of F:

$$F_1, F_2 = \frac{-896{,}854{,}372.3 \pm \sqrt{(896{,}854{,}372.3)^2 - 4(-916{,}120{,}132.9)(-103{,}049{,}073.4)}}{2(-916{,}120{,}132.9)}$$

$= 0.8460, 0.1330$

Step 6: Use F_1 and F_2 to determine two values of T:

$$T_1 = \frac{739.1733}{2((852.90)(0.8460) - 417.4805)} = 1.2154$$

$$T_2 = \frac{739.1733}{2((852.90)(0.1330) - 417.4805)} = -1.2154$$

Since T_1 is positive, $T^* = T_1 = 1.2154$ and $F^* = F_1 = 0.8460$.

Step 7: Determine the values of the N_i^*s:

$$N_1^* = \frac{(10)(1.2154)[0.75 + (0.25)(0.8460)]}{\sqrt{2(120)}}\sqrt{\frac{15.00}{0.25}} = 5.8432$$

Similarly, $N_2^* = 1.7529$, $N_3^* = 4.6365$, and $N_4^* = 6.9137$.

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4.3 Integerizing the continuous optimum solution

Both Banerjee et al. [1] and Sharma and Singh [3] note that the values of the N_is should be integers, although the values determined from (11) will probably not be. There are two basic reasons for requiring integer values. First, for the sake of convenience and logical ordering, it makes no sense to allow a smaller (fractional) component order as the last order of each final-product production cycle. Second, if the orders for a component are always for the same quantity, but a non-integer value for N_i means that part of the last component order during one production cycle for the final product is carried over to the beginning of the next production cycle, then, as noted in Banerjee et al., “the resulting average inventory expression and the cost function become almost intractably complex.” The minimum value of (10) is then a lower bound on the optimal cost rather than the actual minimum.

While it is possible to solve for the optimal values of T, F, and the N_is by using non-linear mixed integer programming, it is easy to get very good solutions with integer values for the N_is by using the integerization heuristic described in Banerjee et al. [1], as follows:

Step 1: Find values for T^* and F^*. Substitute these values repeatedly into (11) to get values for the N_is. If all the N_is are integers, the optimal solution has been found.

Step 2: Fix any N_is with integer values at those values. Beginning with the non-integer N_i with the lowest index, let $\underline{N_i}$ be the largest integer less than N_i and let $\overline{N_i}$ be the smallest integer greater than N_i. (I.e., $\underline{N_i}$ and $\overline{N_i}$ "surround" N_i in Banerjee et al.'s terminology.) Successively substitute $\underline{N_i}$ and $\overline{N_i}$, along with any N_is already fixed at integer values and the remaining non-integer N_is into the objective function, (10). Fix the value of N_i for this index i to be the surrounding value that gives the lower objective function value. Repeat Step 2 until all components have had their value of N_i fixed as an integer. (Note: If any N_i is less than 1, its integerized value must be 1.)

Example – Part 2: Integerizing the N_is from Part 1 of the Example in Section 4.2

In Part 1 of the Example in Section 4.2 we determined that the optimal non-integer values of T, F, and the N_is are: $T^* = 1.2154$, $F^* = 0.8460$, $N_1 = 5.8432$, $N_2 = 1.7529$, $N_3 = 4.6365$, and $N_4 = 6.9137$. All the N_is are non-integer. The value of the objective function is 907.50. Table 1 summarizes the implementation of the integerization heuristic. The successive repetitions of Step 2 are as follows:

N_1: The surrounding values for 5.8432 are 5 and 6. Since the cost with $N_1 = 6$ is lower than the cost with $N_1 = 5$, fix $N_1 = 6$.

N_2: The surrounding values for 1.7529 are 1 and 2. Since the cost with $N_1 = 6$ and $N_2 = 2$ is lower than the cost with $N_1 = 6$ and $N_2 = 1$, fix $N_2 = 2$.

N_3: The surrounding values for 4.6365 are 4 and 5. Since the cost with $N_1 = 6$, $N_2 = 2$, and $N_3 = 5$ is lower than the cost with $N_1 = 6$, $N_2 = 2$, and $N_3 = 4$, fix $N_3 = 5$.

N_4: The surrounding values for 6.9137 are 6 and 7. Since the cost with $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 7$ is lower than the cost with $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 6$, fix $N_4 = 6$.

The "optimal" solution is then: $T = 1.2154$, $F = 0.8460$, $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 7$, with a cost of 907.50.

****

Two questions may come to mind about this procedure.

1. Does integerizing the N_is in this order (i.e., lexicographically) give the best surrounding integer solution?

   While we cannot prove that this procedure gives the optimal integer solution or even the best surrounding integer solution, we can say that it gave the best surrounding integer solution for each of the 96 test problems discussed in Section 6.

2. In this example, simply rounding off each of the non-integer N_is to the nearest integer value would have given the same integer solution. Is this always the case?

   The answer is "No." Again referring to the 96 test problems in Section 6, we found that rounding the non-integer values off gave the best surrounding integer solution in all cases except one, in which one of the N_i values found by rounding was different from its value in the best surrounding integer solution. However, the costs of the two solutions were virtually identical. In addition, for each of those 96 test problems, we found the costs for all the surrounding solutions and determined the ratio of the largest of those costs to the smallest (which was, as noted, the cost of the surrounding solution identified by the heuristic integerization procedure). The average worst/best cost ratio for the 96 test cases was 1.00005 and the largest ratio was 1.0003. Thus, selecting which surrounding solution to use by any other method would have virtually no impact on the cost of the solution.

5. Three heuristics

Although finding the "optimal" solution to the single-product EPQ with partial backordering and coordinated planning for its components is not difficult, there are other approaches to the problem that are either easier to implement or have other desirable characteristics. In this section we consider three heuristics: A) using Pentico et al.'s [4] optimal solution procedure for the EPQ with partial backordering and no components (EPQ-PBO) to determine the values of T and F, combined with the use of (11) to determine the initial values of the N_is, which will then be integerized; B) using the same integerized value of N for all components; and C) using lot-for-lot (L4L) ordering of the components. In Section 6 we describe a computational study done to evaluate the performance of these heuristics and discuss the results.

5.1 Heuristic A: using Pentico et al.'s EPQ-PBO model

Pentico et al.'s [4] model was developed for controlling the inventory of a single item that meets the conditions of the classic EPQ with full backordering except that only a fraction β of stockouts will be backordered. As in Section 3, it uses T as its primary decision variables. Since it does not control the coordinated production of a final product and its components, it does not include equations for the N_is. This heuristic uses Pentico et al.'s equations for T and F and then uses those values in (11) to get the optimal values of the N_is for given T and F. Since the values of the N_is will probably not be integers, it then uses the procedure in Section 4.3 to integerize the N_is.

Heuristic A: Using Pentico et al.'s [4] equations for T and F

Step 1: Use the EOQ-PBO equations to get values for T and F:

$$T_A^* = \sqrt{\frac{2C_o}{DC_h'}\left[\frac{C_h' + \beta C_b'}{\beta C_b'}\right] - \frac{[(1-\beta)C_l]^2}{\beta C_h' C_b'}} \tag{22}$$

$$F_A^* = F(T_A^*) = \frac{(1-\beta)C_l + \beta C_b' T_A^*}{T_A^*(C_h' + \beta C_b')} \tag{23}$$

where all the parameters are defined as in Section 3.

Step 2: Use (11), repeated as (24), to determine the optimal values for the N_is given T and F:

$$N_i = \frac{DT_A^*[\beta + (1-\beta)F_A^*]}{\sqrt{2P}}\sqrt{\frac{C_{hi}'}{C_{oi}}} \tag{24}$$

Step 3: Use the procedure in Section 4.3 to integerize the N_is.

Example – Part 3: Using Heuristic A

We use the same parameter values as in Parts 1 and 2 of the Example in Section 4.

Step 1: Determine the values of T_A^* and F_A^*:

$$T_A^* = \sqrt{\frac{2(475)}{(10)(87.0833)}\left[\frac{(87.0833) + (0.75)(111.3281)}{(0.75)(111.3281)}\right] - \frac{[(0.25)(296.875)]^2}{(0.75)(87.0833)(111.3281)}} = 1.2129$$

$$F_A^* = \frac{(0.25)(296.875) + (0.75)(111.3281)(1.2129)}{(1.2129)[87.0833 + (0.75)9111.3281)]} = 0.8482$$

Step 2: Determine the values of the N_is using (24):

$$N_1 = \frac{(10)(1.2129)[0.75 + (0.25)(0.8482)]}{\sqrt{2(120)}}\sqrt{\frac{15.00}{0.25}} = 5.8343$$

Similarly, we get $N_2 = 1.7503$, $N_3 = 4.6295$, and $N_4 = 6.9033$.

Following the same process used in Example – Part 2 in Section 4.3, the integerized values of the N_is are determined to be: $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, $N_4 = 7$. Heuristic A's cost is 907.56, which is the same as the cost of the best integer solution found in Part 2 of the Example in Section 4.3.

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5.2 Heuristic B: using the same N for all components

The concept of this heuristic is to fully coordinate the production of the components by ordering them all at the same time. This heuristic may be particularly of interest if all the components are purchased from the same supplier so that they may be shipped in together. As noted earlier, using the EOQ assumptions instead of the EPQ assumptions for the components requires simply replacing C'_{hi} by C_{hi} in the equations.

Dropping the subscripts on the *N*s in the total cost Eq. (10) in Section 3.2.3 gives the cost equation for Heuristic B:

$$\Gamma_B(T,F,N) = \frac{C_o + N\sum_{i=1}^{m} C_{oi}}{T} + C'_h \frac{DTF^2}{2} + \frac{TD^2[\beta + (1-\beta)F]^2}{2PN}\sum_{i=1}^{m} C'_{hi} + C'_b \frac{\beta DT(1\text{-}F)^2}{2}$$

$$+ C_l D(1-\beta)(1-F) \qquad (25)$$

Setting the partial derivative of $\Gamma_B(T,F,N)$ with respect to N equal to 0 gives:

$$N = \frac{DT[\beta + (1-\beta)F]}{\sqrt{2P}}\sqrt{\frac{\sum_{i=1}^{m} C'_{hi}}{\sum_{i=1}^{m} C_{oi}}} \qquad (26)$$

Following the process used in Section 4.2 for developing the optimal solution for the basic problem gives:

$$\Gamma_B^*(T,F) = \frac{C_o}{T} + \frac{D[\beta + (1-\beta)F]\sqrt{\sum_1^m C'_{hi}\sum_1^m C_{oi}}}{\sqrt{2P}} + C'_h \frac{DTF^2}{2}$$

$$+\frac{D[\beta+(1-\beta)F]\sqrt{\sum_1^m C'_{hi}\sum_1^m C_{oi}}}{\sqrt{2P}} + C'_b\,\frac{\beta DT(1\text{-}F)^2}{2} + C_l D(1-\beta)(1-F) \qquad (27)$$

This can be expressed in the same form as in (12):

$$\Gamma^*_B\,(T,F) = \frac{G_o}{T} + TR(F) + Q(F) \qquad (28)$$

where:

$$R(F) = G_1F^2 - 2G_2F + G_2 \text{ and } Q(F) = G_3F + G_4 \qquad (29)$$

with:

$$G_0 = C_0 \qquad G_1 = \frac{D(C'_h + \beta C'_b)}{2} \qquad G_2 = \frac{\beta C'_b D}{2} \qquad (30)$$

$$G_3 = \frac{2D(1-\beta)\sqrt{\sum_1^m C'_{hi}\sum_1^m C_{oi}}}{\sqrt{2P}} - C_l D(1-\beta) \qquad G_4 = \frac{2D\beta\sqrt{\sum_1^m C'_{hi}\sum_1^m C_{oi}}}{\sqrt{2P}} + C_l D(1-\beta)$$

Lemmas 1 and 2 of Section 4.1 follow exactly, as do Corollaries 1 and 2. Lemma 3 also follows exactly as in Section 4.1, with no changes since the equations for *a, b, c* and *T** are expressed in terms of the *G*s, not in terms of their definitions in terms of the problem parameters.

Steps 1-7 for finding the solution with the added requirement that all *N*s have the same value are identical to Steps 1-7 in Section 4.2 except that (30) is used in Step 1 and (26) is used once in Step 7 instead of (11) being used m times. Then add Step 8 to integerize *N*.

Step 8: If *N* is not an integer, compare the costs of using the integer values surrounding *N* to determine the integerized value of *N*.

Example – Part 4: Using Heuristic B

We use the same parameter values as in the previous parts of the Example.

Step 1: Compute the values of G_0, G_1, G_2, G_3, and G_4 from (30).

G_0 = 475 G_1 =10(87.0833 + (0.75)(111.3281))/2 = 852.90

G_2 = (0.75)(111.3281)/2 = 417.4805

$$G_3 = \frac{2(10)(0.25)\sqrt{(48.2667)(2.6)}}{\sqrt{2(120)}} - (296.875)(10)(0.75) = -738.5719$$

$$G_4 = \frac{2(10)(0.75)\sqrt{(48.2667)(2.6)}}{\sqrt{2(120)}} + (296.875)(10)(0.25) = 753.0342$$

Step 2: $G_3 - 2\sqrt{G_0 G_2} = -738.5719 - 2\sqrt{(475)(417.4805)} = -1629.20$, which is not > 0, so go to Step 3.

Step 3: $G_3 + 2\sqrt{G_0(G_1 - G_2)} = -738.5719 + 2\sqrt{(475)(852.90 - 417.4805)} = 170.98$, which is not < 0, so go to Step 4.

Step 4: Compute the values of *a, b*, and *c*:

$$a = (852.90)(-738.5719)^2 - 4(475)(852.90)^2 = -916{,}878{,}097.7$$

$$b = 8(475)(852.90)(417.4805) - 2(417.4805)(-738.5719)^2 = 897{,}596{,}397.4$$

$$c = (417.4805)(-738.5719)^2 - 4(475)(417.4805)^2 = -103{,}420{,}085.9$$

Step 5: Use the quadratic formula to determine two values of *F*:

$$F_1, F_2 = \frac{-897596397.4 \pm \sqrt{(897596397.4)^2 - 4(-916878097.7)(-103420085.9)}}{2(-916878097.7)}$$

$$= 0.8456, 0.1334$$

Step 6: Use F_1 and F_2 to determine two values of *T*:

$$T_1 = \frac{738.5719}{2((852.90)(0.8456) - 417.4805)} = 1.2159$$

$$T_2 = \frac{738.5719}{2((852.90)(0.1334) - 417.4805)} = -1.2159$$

Since T_1 is positive, $T_B^* = T_1 = 1.2159$ and $F_B^* = F_1 = 0.8456$.

Step 7: Substituting the values of T_B^* and F_B^* into Eq. (26) gives:

$$N = \frac{(10)(1.2159)[0.75 + (0.25)(0.8456)]}{\sqrt{2(120)}} \sqrt{\frac{28.2667}{2.6}} = 4.6379$$

Step 8: The integers surrounding $N = 4.6379$ are 4 and 5. The cost with $N = 4$ is 910.11 and the cost with $N = 5$ is 911.12, so $N = 4$.

The solution using Heuristic B is $T_B^* = 1.2159$, $F_B^* = 0.8456$, and $N = 4$. Heuristic B's cost is 910.11, which is 1.0029 times as large as the cost of the best integer solution found in Part 2 of the Example in Section 4.3.

****

In Sections 3 and 4 we developed a model and procedure for finding possibly different N_is for each component. In Heuristic B we modified the model and procedure to require that all components be ordered at the same time. In some situations it may be desirable to use an intermediate approach in which components are grouped into sets, with each component in a set being ordered at the same time, but different sets being ordered at different times. In Appendix B we show how our basic model can be modified to develop an optimal production plan for this situation.

5.3 Heuristic C: lot-for-lot

Banerjee et al. [1] mention lot-for-lot (L4L) ordering as a special case of the problem considered here. Sharma and Singh [3], however, citing work by Anwar and Nagi [14], note that, at least in the case of complex assemblies, it may not be the best JIT strategy. However, since L4L is mentioned fairly often in the literature as a possible method for scheduling the production of components for a finished product, we shall consider it here.

Our starting point for developing the equations for the L4L heuristic is to note that this is a special case of Heuristic B in which $N = 1$ for all components. Thus, our first step is to set $N = 1$ in (25):

$$\Gamma_C(T,F) = \frac{C_o + \sum_{i=1}^{m} C_{oi}}{T} + C_h^{'}\frac{DTF^2}{2} + \frac{TD^2[\beta+(1-\beta)F]^2}{2P}\sum_{i=1}^{m} C_{hi}^{'} + C_b^{'}\frac{\beta DT(1\text{-}F)^2}{2}$$

$$+ C_l D(1-\beta)(1-F) \tag{31}$$

This can be expressed in the same form as in (12):

$$\Gamma_C^{*}(\mathrm{T,F}) = \frac{G_o}{T} + TR(F) + Q(F) \tag{32}$$

where:

$$R(F) = G_1F^2 + G_2F + G_3 \text{ and } Q(F) = G_4F - G_4 \tag{33}$$

with:

$$G_0 = C_0 + \sum_{i=1}^{m} C_{oi} \qquad G_1 = \frac{D(C_h^{'} + \beta C_b^{'})}{2} + \frac{D^2(1-\beta)^2\sum_{i=1}^{m} C_{hi}^{'}}{2P} \tag{34}$$

$$G_2 = \frac{2\beta(1-\beta)D^2\sum_{i=1}^{m} C_{hi}^{'}}{2P} - \frac{2\beta C_b^{'}D}{2} \qquad G_3 = \frac{\beta C_b^{'}D}{2} + \frac{D^2\beta^2\sum_{i=1}^{m} C_{hi}^{'}}{2P} \qquad G_4 = -C_l D(1-\beta)$$

Note that $R(F)$ and $Q(F)$ are similar, but not identical, to the comparable equations in Section 4.1 for the optimal solution and Section 5.2 for Heuristic B. Also, the equations for the Gs in (34) are different from the ones in those two sections. This has some implications for determining the values of a, b, and c and, based on them, the values of T_C^{*} and F_C^{*}.

As in the proof of Lemma 2 in Section 4.1:

$$T_C^{*} = T_C^{*}(F) = \sqrt{\frac{G_0}{R(F)}} \tag{35}$$

$$\Gamma_{C,T}^{*}(F) = \Gamma_C^{*}(T_C^{*}, F) = 2\sqrt{G_0R(F)} + Q(F) \tag{36}$$

and

$$\Gamma^{*}_{C,T}{}'(F) = \sqrt{G_0}\,\frac{R'(F)}{(R(F))^{1/2}} + Q'(F)\,. \tag{37}$$

Taking the derivatives of *R(F)* and *Q(F)* with respect to *F* gives $R'(F) = 2G_1F + G_2$ and $Q'(F) = G_4$. This leads to:

$$\Gamma^{*}_{C,T}(F) = \frac{\sqrt{G_0}\,(2G_1F + G_2)}{\sqrt{G_1F^2 + G_2F + G_3}} + G_4 \tag{38}$$

Substituting $F = 0$ into (38), the equivalent of Corollary 1 is:

$F^* = 0$ if and only if $G_2\sqrt{G_0/G_3} + G_4 > 0$.

Substituting $F = 1$ into (38), the equivalent of Corollary 2 is:

$F^* = 1$ if and only if $\dfrac{\sqrt{G_0}\,(2G_1 + G_2)}{\sqrt{G_1 + G_2 + G_3}} + G_4 < 0$.

Lemma 3 also follows as in Section 4.1, with changes in the equations for *a, b, c* and T_C^* to reflect the changes in the equations for *R(F)* and *Q(F)*. Using the equations for *R(F), R'(F), Q(F)*, and *Q'(F)* given above, the equations for *a, b*, and *c* are:

$$a = G_1G_4^2 - 4G_0G_1^2 \qquad b = G_2G_4^2 - 4G_0G_1G_2 \qquad c = G_3G_4^2 - G_0G_2^2 \tag{39}$$

and *T* is determined from:

$$T = \frac{-G_4}{2G_1F + G_2} \tag{40}$$

The steps to find the solution with the added requirement that all *N*s are equal to 1 are:

Step 1: Compute the values of G_0, G_1, G_2, G_3, and G_4 from (34).

Step 2: If $G_2\sqrt{G_0/G_3} + G_4 > 0$, set $T = \infty$ and $F = 0$. The product should not be produced. If not, go to Step 3.

Step 3: If $\dfrac{\sqrt{G_0}(2G_1+G_2)}{\sqrt{G_1+G_2+G_3}}+G_4<0$, set $F = 1$ and $T = \sqrt{2C_o/(DC_h^{'})}$. The product should be produced with no backordering. If not, go to Step 4.

Step 4: Compute the values of *a, b*, and *c* from (39).

Step 5: Use the quadratic formula to determine two values of *F*:

$$F_1,\ F_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Step 6: Use F_1 and F_2 to determine two values of T from:

$$T = \frac{-G_4}{2G_1F+G_2}$$

Set $T_C^* = T_1$ or T_2, whichever is positive. Set F_C^* = the value of F that gave T_C^*.

Example – Part 5: Using Heuristic C

We use the same parameter values as in the previous parts of the Example.

Step 1: Compute the values of G_0, G_1, G_2, G_3, and G_4.

$$G_0 = 475 + 2.60 = 477.60$$

$$G_1 = \frac{(10)(87.0833+(0.75)(111.3281))}{2}+\frac{(10)^2(0.25)^2(48.2667)}{2(120)} = 854.1541$$

$$G_2 = \frac{2(0.75)(0.25)(10)^2(48.2667)}{2(120)}-\frac{2(0.75)(111.3281)(10)}{2} = -827.4193$$

$$G_3 = \frac{(0.75)(111.3281)9100}{2}+\frac{9100^2(0.75)^2(48.2667)}{2(120)} = 428.7930$$

$$G_4 = -(296.875)(10)(0.25) = -742.1875$$

Step 2: $G_2\sqrt{G_0/G_3}+G_4 = -827.4193\sqrt{477.60/428.793}+(-742.1875) = -1615$, which is not > 0, so go to Step 3.

Step 3: $\dfrac{\sqrt{G_0}\,(2G_1+G_2)}{\sqrt{G_1+G_2+G_3}}+G_4 = \dfrac{\sqrt{477.60}\,(2(854.1541)+(-827.4193))}{\sqrt{854.1541+(-827.4193)+428.7930}}+(-742.1875)=159.79,$

which is not < 0, so go to Step 4.

Step 4: Compute the values of *a, b*, and *c*.

$a = (854.1541)(-742.1875)^2 - 4(477.6)(854.1541)^2 = -923{,}283{,}903.44$

$b = (-827.4193)(-724.1875)^2 - 4(477.6)(854.1541)(-827.4193) = 894{,}385{,}348.23$

$c = (428.793)(-742.1875)^2 - (477.6)(-827.4193)^2 = -90{,}778{,}478.75$

Step 5: Use the quadratic formula to determine two values of F:

$$F_1, F_2 = \frac{-894385348.23 \pm \sqrt{(894385348.23)^2 - 4(-923283903.44)(-90778478.75)}}{2(-923283903.44)}$$

$= 0.8535,\ 0.1152$

Step 6: Use F_1 and F_2 to determine two values of T:

$$T_1 = \frac{742.1875}{2((854.1541)(0.8535)-827.4193)} = 1.1769$$

$$T_2 = \frac{742.1875}{2((854.1541)(0.1152)-827.4193)} = -1.1769$$

Since T_1 is positive, $T_C^* = T_1 = 1.1769$ and $F_C^* = F_1 = 0.8535$.

The solution using Heuristic C is $T_C^* = 1.1769$, $F_C^* = 0.8535$, with $N_i = 1$ for all components. Heuristic C's cost is 920.35, which is 1.0142 times as large as the cost of the best integer solution found in Part 2 of the Example in Section 4.3.

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6. Computational study

To gain some insights into how well Banerjee et al.'s [1] integerization procedure works for this problem and how well the three heuristics described in Section 5 might perform, we identified six situational characteristics or factors that we thought might make a difference in performance, selected reasonable and reasonably different values for them, and ran a series of test cases. In Section 6.1 we describe the characteristics and their selected values. In Section 6.2 we summarize the results for the test cases.

6.1 Situational characteristics and levels for the study

We identified six factors that we thought might make a difference in how well the integerization procedure would work and how well the three heuristics would work. We selected two levels for five factors and three levels for the sixth. The characteristics and their selected levels are:

- Number of components: 2 or 8
- Ratio of C_o to C_h for the final product: 0.5 or 5.0
- Ratio of C_l to C_b for the final product: 1.25 or 2.50
- Ratio of P to D for the final product : 3 or 10
- Value of β: 0.75 or 0.95
- The mix of desired ordering frequencies for the components: all monthly, all weekly, or half monthly and half weekly. This was included to look at the issue of using JIT control of the components.

In addition to these factors that were varied to see what their effects might be, we selected values for other characteristics that would be held constant to the extent possible. These characteristics and their values or the ways in which their values were determined are:

- Component holding costs: Costs per year to hold a unit of a component are in the range of approximately \$1,000 to \$2,000, which might be appropriate if the basic unit of demand for the final product was 100 or 1,000 "real" units.

- The final product's holding cost per unit per year is 1.25 times the sum of its components' holding costs to allow for value added in producing the final product.
- The final product's backordering cost per unit per year is 1.25 times the unit's holding cost per year.
- The final product's demand per year is a little smaller than the largest value of D that will keep β^* for that product less than 0.75 if components are ignored and planning for partial backordering of the final product alone is considered. This was done to avoid having "no backordering" be optimal.
- The production rates per year for the components are 4 times the production rate for the final product for half of the components and 10 times for the other half.
- The ordering/set-up costs for the components are set at values that will give approximately the desired frequency of ordering (monthly or weekly) for each component. The actual frequencies used were randomly adjusted to be within approximately 10 percent of the desired frequencies, with about half being higher and half being lower.

6.2 The experimental results

With five factors each having two possible levels and one having three possible levels, there are 96 combinations or test cases. All 96 cases were solved with the continuous optimization procedure in Section 4.2, with the cost of this solution being the standard against which each of the other solution procedures was compared. The continuous solution was then integerized, as described in Section 4.3. Each problem was also solved using the three heuristics described in Section 5. The performance measure used is the ratio of a procedure's cost to the cost obtained from the continuous optimization's solution.

Table 2 gives the average and maximum ratios for each of the solution methods. It also shows for how many of the 96 cases that method gave a solution with a cost ratio less than 1.00005, which means that it is essentially as good as the continuous optimum solution.

First, we see that the integerization procedure found an integer solution essentially as good as the continuous optimum in all 96 cases. Further, we see that Heuristic A – using Pentico et al.'s [4] EOQ-PBO model to determine the values of T and F and then using the integerization procedure on the N_is – did almost as well, with an average cost ratio of 1.00001, a maximum cost ratio of 1.0001, and an essentially optimal solution in 93 of the 96 cases.

Somewhat surprisingly, Heuristic B – finding the optimal solution with the added requirement that all components use the same integerized N – did quite well. While it found an essentially optimum solution in only 39 of the 96 cases, the average cost ratio was less than 1.001, or less than 0.1 percent above the optimum, and the maximum ratio was 1.0069. Although we cannot generalize the results of this study to the broader set of possible applications of these rules, they do suggest that fully coordinating the production or receipt of components should increase costs very little.

As might be expected, Heuristic C – ordering components on a lot-for-lot basis relative to the orders for the final product – did not do well. The average cost ratio was over 1.10, with a maximum of 1.2169 and a minimum of 1.0235. In no case did it give an essentially optimal solution.

Breaking the 96 cases down into subsets based on the values of the factors, we found, as might be expected, that there was very little difference in the average or maximum cost ratios for any of the subgroupings for the integerization of the continuous solution, Heuristic A, or Heuristic B. To a large extent, the same is true for Heuristic C, with some, but not a lot of, difference in the averages or maximums for the subgroupings by the number of components, the C_o/C_h ratio, the C_l/C_b ratio, or the mix of desired N_i values. There were, however, much larger values for the average (1.16721), maximum (1.2169), and minimum (1.1162) ratios with $P/D = 3$ versus $P/D = 10$ (average = 1.0434, maximum = 1.0807, minimum = 1.0235) and somewhat larger values for the average (1.12265) and maximum (1.2169) ratios with $\beta = 0.95$ versus $\beta = 0.75$ (average = 1.08491, maximum = 1.1560).

7. Conclusion

We have extended the models by Banerjee et al. [1] and Sharma and Singh [3] for jointly planning the production of a final product subject to the conditions of an EPQ model and the production or purchasing of its components to the case where the final product is subject to partial backordering of unfilled demand. Beginning with Pentico et al.'s [4] model for the EPQ with partial backordering that finds the optimal values for T, the time between orders, and F, the fill rate or percentage of demand filled from stock, we incorporated the costs of ordering and carrying inventory of the components as well as those of the final product. We used classical calculus methods to find an optimal solution to the relaxed version of the problem that did not require integer numbers of component orders per order cycle of the final product and then used the heuristic integerization procedure developed by Banerjee et al. [1] to find an integer solution. A numerical study with 96 test cases based on considering six situational characteristics that might affect the performance of the model found that the cost of the integerized solution was, in all cases, within 0.005 percent of the lower bound on cost given by the relaxed continuous solution.

In addition to the integerized continuous optimum solution procedure, we proposed three heuristic solution methods: A) using Pentico et al.'s [4] model to determine T and F and then using those values in the equation to determine the optimal values for the numbers of orders for each component given *T* and *F*, B) requiring that all components be ordered the same number of times and determining the values of *T*, *F*, and *N* that would minimize the cost, and C) using lot-for-lot ordering of the components and determining the optimal values of *T* and *F* with that requirement. Using the same set of 96 test cases that were used to evaluate the performance of the optimum-seeking model, we found that Heuristic A performed almost as well as the optimization procedure, Heuristic B performed extremely well, with an average increase over the lower bound of less than 0.01 percent, and Heuristic C, as expected, did poorly, with an average increase over the lower bound of more than 10 percent.

Appendix A: Incorporating less-than-perfect yield for the final product

One of the two major differences between the model developed here and that of Sharma and Singh [3] is that they included the possibility that the yield from the final product's production would not be 100 percent. It is straightforward to add that to our model.

First, we add one new parameter to the model: y = the fraction of the final product's production process that is not defective. The revised version of (10), found by replacing P by yP everywhere it appears, including the definitions of $C_h^{'}$, $C_b^{'}$, and $t_2 + t_3$ used in defining Q_i and $\bar{I}_i$, except in the definition of $C_{hi}^{'}$, is:

$$\Gamma(T,F,N_1,\ldots,N_m) = \frac{C_o + \sum_{i=1}^{m} N_i C_{oi}}{T} + C_h^{'}\frac{DTF^2}{2} + \frac{TD^2[\beta+(1-\beta)F]^2}{2Py^2}\sum_{i=1}^{m}\frac{C_{hi}^{'}}{N_i} + C_b^{'}\frac{\beta DT(1\text{-}F)^2}{2}$$
$$+ C_l D(1-\beta)(1-F) \qquad \text{(A-1)}$$

The revised version of (11) in Lemma 1 is then:

$$N_i^* = \frac{DT[\beta + (1-\beta)F]}{y\sqrt{2P}} \sqrt{\frac{C_{hi}^{'}}{C_{oi}}} \quad \text{(A-2)}$$

and the revised equations (set (13)) for G_3 and G_4 in Lemma 2 are:

$$G_3 = \frac{2D(1-\beta)}{y\sqrt{2P}} \sum_1^m \sqrt{C_{hi}^{'} C_{oi}} - C_l D(1-\beta) \qquad G_4 = \frac{2D\beta}{y\sqrt{2P}} \sum_1^m \sqrt{C_{hi}^{'} C_{oi}} - C_l D(1-\beta) \quad \text{(A-3)}$$

The rest of the development in Section 4.1 is the same, and the procedures for finding the optimal continuous solution and integerizing it are the same.

Appendix B: Joint ordering of multiple components

In the basic model developed in Section 3 and 4 we assumed that the components would be ordered independently, so there are m possibly different N_is. In Section 5.2 we assumed that all the components would be ordered at the same time, so there is a single N. Here we assume that the components are grouped into distinct sets, with all the components in a specific set being ordered at the same time, possibly from the same supplier, but the different sets may be ordered at different times and different frequencies.

It is often the case in joint ordering that the "fixed" cost of placing an order is broken down into two parts: the "header" fixed cost of placing the order for the group and the "line" fixed cost associated with each item in the order. To allow for this in the model, we define additional parameters:

g = the number of groups of items, the items in each group being jointly ordered

A_k = the set of components i that are in group k

C_{fk} = the fixed header cost to place an order for group k

C_{oi} = the fixed line cost to place an order for component i (as in the original model)

$\hat{C}_{ok} = C_{fk} + \sum_{i\varepsilon A_k} C_{oi}$ = the total fixed cost of placing an order for the components in group k

N_k = the number of times group k is ordered for each production cycle of the final product

To transform (10), to cost function to be minimized, to recognize this change, we first change (7), the equation for Q_i, the amount of component i used in each production cycle for the final product, and (8), the equation for $\bar{I}_i$, the average inventory for component i, by replacing N_i by N_k. The cost function to be minimized is, then:

$$\Gamma(T,F,N_1,\ldots,N_g) = \frac{C_o + \sum_{k=1}^{g} N_k \hat{C}_{ok}}{T} + C_h^{'} \frac{DTF^2}{2} + \frac{TD^2[\beta + (1-\beta)F]^2}{2P} \sum_{k=1}^{g} \frac{\sum_{i\varepsilon A_k} C_{hi}^{'}}{N_k}$$

$$+ C_b^{'} \frac{\beta DT(1\text{-}F)^2}{2} + C_l D(1-\beta)(1-F). \qquad \text{(B-1)}$$

Lemma 1 in Section 4.1 still holds with a new version of (11):

$$N_k^* = \frac{DT[\beta + (1-\beta)F]}{\sqrt{2P}} \sqrt{\frac{\sum_{i\varepsilon A_k} C_{hi}^{'}}{\hat{C}_{ok}}}\,. \qquad \text{(B-2)}$$

The rest of the theoretical development in Section 4.1 holds with the replacement of G_3 and G_4 in (13) by:

$$G_3 = \frac{2D(1-\beta)}{\sqrt{2P}} \sum_{k=1}^{g} \sqrt{\hat{C}_{ok} \sum_{i\varepsilon A_k} C_{hi}^{'}} - C_l D(1-\beta) \qquad \text{(B-3)}$$

$$G_4 = \frac{2D\beta}{\sqrt{2P}} \sum_{k=1}^{g} \sqrt{\hat{C}_{ok} \sum_{i\varepsilon A_k} C_{hi}^{'}} - C_l D(1-\beta) \qquad \text{(B-4)}$$

With these changes, the process given in Section 4.1 for finding the optimal values of T and F and the process for integerizing the N_ks described in Section 4.2 are the same.

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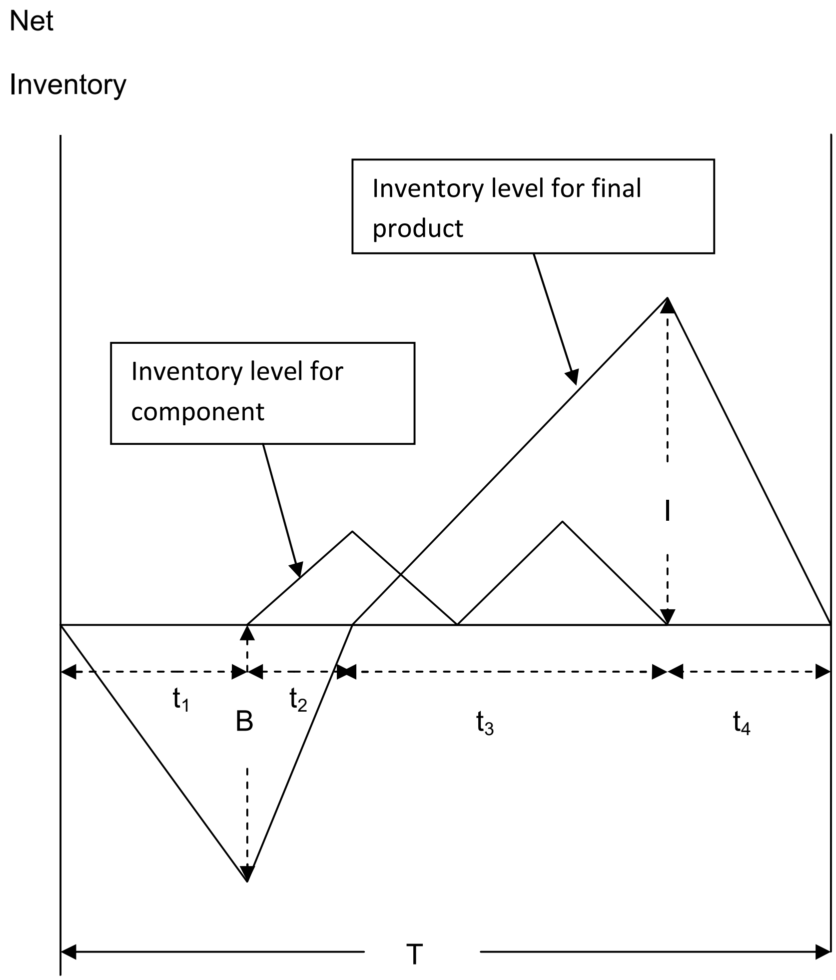

Graphical representation of one production cycle for the final product
with two production cycles for a component

Figure 1

Table 1

Integerizing the N_i Values for Example Problem – Part 1

	N_1	N_2	N_3	N_4	Cost
Continuous Solution	5.8432	1.7529	4.6365	6.9137	907.50
Fix N_1	5	1.7529	4.6365	6.9137	907.53
	6*	1.7529	4.6365	6.9137	907.50*
Fix N_2	6	1	4.6365	6.9137	908.44
	6	2*	4.6365	6.9137	907.55*
Fix N_3	6	2	4	6.9137	907.57
	6	2	5*	6.9137	907.56*
Fix N_4	6	2	5	6	907.58
	6	2	5	7*	907.56*

Table 2

Summary of Ratios of Procedure Costs to Cost of Continuous Optimum Solution

	Integerized	Heuristic A	Heuristic B	Heuristic C
Average Ratio	1.00000	1.00001	1.00088	1.10378
Maximum Ratio	1.0000	1.0001	1.0069	1.2169
Minimum Ratio	1.0000	1.0000	1.0000	1.0235
No. < 1.00005	96	93	39	0