# DIFFERENCES OF COMPOSITION OPERATORS

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Abstract. We consider composition operators acting on the Hardy space and all the standard weighted Bergman spaces, and present a sufficient condition for the difference of two such operators to be compact. The proof technique lends itself to estimating norms of composition operator differences, and this extension is used to draw some conclusions about the topological structure of the set of composition operators. In particular, a conjecture of Joel Shapiro and Carl Sundberg is answered negatively.

Let $D$ be the open unit disk in the complex plane, $B$ a Banach space of analytic functions on $D$, and $\mathcal{C}(B)$ the set of bounded composition operators with analytic symbol acting on $B$. The component structure of $\mathcal{C}(H^p)$ in the topology induced by the operator norm was first studied by Earl Berkson in [B]; subsequent investigation extended to the component structure of $\mathcal{C}(B)$ for a variety of different Banach spaces $B$ in a variety of different topologies (see [SS],[M],[MOZ],and [HJM].) In the setting of the norm topology on $\mathcal{C}(H^2)$, Joel Shapiro and Carl Sundberg conjecture that two operators $C_\phi$ and $C_\psi$ lie in the same component if and only if they have compact difference (see [SS]). In [S], Jonathan Shapiro investigates compact composition operator differences in the setting of the Hardy space using Aleksandrov measures, and in [G], Tom Goebeler characterizes the same for composition operators acting on mixed Hardy spaces. Here we use different techniques to explore compact difference for composition operators acting on a range of spaces, including the Hardy space and all the standard weighted Bergman spaces. We provide a sufficient condition for compactness in terms of weighted composition operators acting between these spaces, and use this condition to provide a counterexample to the Shapiro-Sundberg conjecture.

Recall that the Hardy space $H^2$ consists of those functions $f$, analytic on the unit disk, which satisfy $\|f\|_{H^2}^2 := \lim_{r\to 1}\int_{\partial D}|f(r\zeta)|^2 d\sigma(\zeta) < \infty$, where $\sigma$ is normalized Lebesgue measure on the boundary of the disk. For $\alpha > -1$, the standard weighted Bergman space $A^2_\alpha$ is the set of functions analytic on the disk with $\|f\|^2_{A^2_\alpha} := \int_D |f(z)|^2 d\lambda_\alpha(z) < \infty$, where $d\lambda_\alpha(z)$ is the weighted area measure $\frac{(\alpha+1)}{\pi}(1-|z|^2)^\alpha dA(z)$. Using power series (see [CM]), it is not hard to show that if $f \in A^2_\alpha$, then $f' \in A^2_{\alpha+2}$ with $\|f'\|_{A^2_{\alpha+2}} \leq M_\alpha \|f\|_{A^2_\alpha}$ for some constant $M_\alpha$ that does not depend on $f$. Similarly, if $f \in H^2$, then $f' \in A^2_1$ and $\|f'\|_{A^2_1} \leq M_H \|f\|_{H^2}$ for a constant $M_H$ independent of $f$.

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We will make use of the following, whose proof is a slight modification of the proof of Proposition 3.11 in [CM].

**Proposition 1.** *Let $\mathcal{X}$ and $\mathcal{Y}$ each represent either the Hardy space or a standard weighted Bergman space. Then a finite sum of weighted composition operators $\sum w_i C_{\phi_i}$ is compact from $\mathcal{X}$ to $\mathcal{Y}$ if and only if whenever $\{f_n\}$ is bounded in $\mathcal{X}$ and $f_n \to 0$ uniformly on compact subsets of $D$, then $(\sum w_i C_{\phi_i}) f_n \to 0$ in $\mathcal{Y}$.*

**Theorem 1.** *Let $\phi$ and $\psi$ be analytic self maps of the disk, and define $\phi_s(z) := s\phi(z) + (1-s)\psi(z)$ for $0 \le s \le 1$. Let $w(z)$ denote the bounded analytic function $\phi(z) - \psi(z)$. If the weighted composition operators $wC_{\phi_s} : A^2_{\alpha+2} \to A^2_\alpha$ are uniformly norm bounded in $s$ and, moreover, compact for each $s$, then $C_\phi - C_\psi$ is compact from $A^2_\alpha$ to $A^2_\alpha$. Further, the result holds on the Hardy space, provided the operators $wC_{\phi_s} : A^2_1 \to H^2$ satisfy the given conditions.*

*Proof.* First consider the weighted Bergman spaces. Fix an $f \in A^2_\alpha$ and write

$$(C_\phi - C_\psi)f(z) = f(\phi(z)) - f(\psi(z)) = w(z)\int_0^1 f'(\phi_s(z))ds,$$

whence

$$\begin{aligned}
\|(C_\phi - C_\psi)f\|^2_{A^2_\alpha} &= \int_D |(C_\phi - C_\psi)f(z)|^2 \, d\lambda_\alpha(z) \\
&= \int_D \left| w(z)\int_0^1 f'(\phi_s(z))ds \right|^2 d\lambda_\alpha(z) \\
&\le \int_D \int_0^1 |w(z)f'(\phi_s(z))|^2 \, ds d\lambda_\alpha(z) \\
&= \int_0^1 \|wC_{\phi_s} f'\|^2_{A^2_\alpha} ds.
\end{aligned}$$

If $f_n$ is any bounded sequence in $A^2_\alpha$ converging almost uniformly to zero, then $f_n'$ is a bounded sequence in $A^2_{\alpha+2}$ which converges almost uniformly to zero; the uniform boundedness of the norms of $wC_{\phi_s}$ acting from $A^2_{\alpha+2}$ to $A^2_\alpha$ now permits us to apply Lebesgue's Dominated Convergence Theorem to conclude:

$$\begin{aligned}
\limsup_{n\to\infty} \|(C_\phi - C_\psi)f_n\|^2_{A^2_\alpha} &\le \limsup_{n\to\infty} \int_0^1 \|wC_{\phi_s} f_n'\|^2_{A^2_\alpha} ds \\
&= \int_0^1 \limsup_{n\to\infty} \|wC_{\phi_s} f_n'\|^2_{A^2_\alpha} ds \\
&= 0
\end{aligned}$$

where the last line follows from the compactness of $wC_{\phi_s} : A^2_{\alpha+2} \to A^2_\alpha$ and the "only if" part of Proposition 1. The compactness of $C_\phi - C_\psi : A^2_\alpha \to A^2_\alpha$ now follows from the "if" part of the same.

For the case of the Hardy space, replace integration over $D$ by the integral that defines the Hardy space norm; the same calculations then apply. □

**Remark:** Since the hypothesis of uniform norm boundedness was only used to apply Dominated Covergence in the second line of the last display, it is clear that this assumption can be weakened. Also note that the same techniques can be used to show that $C_{\phi_s} - C_{\phi_t}$ is compact for any $0 \le s \le t \le 1$.

The utility of this condition lies in the fact that the boundedness and compactness of these operators lend themselves to a simple Carleson-measure characterization. Recall that if $\mu$ is a positive, finite Borel measure on the open disk $D$, then $A^2_\alpha \subset L^2(\mu)$ if and only if $\mu$ is an $\alpha-$*Carleson measure*, i.e if and only if

$$\|\mu\|_\alpha := \sup_{S(\zeta,\delta)} \frac{\mu(S(\zeta,\delta))}{\delta^{\alpha+2}} < \infty, \tag{1}$$

where the supremum is over all Carleson sets $S(\zeta,\delta) := \{z \in D : |z-\zeta| < \delta, \zeta \in \partial D\}$. In this case, the inclusion map $I_\alpha : A^2_\alpha \to L^2(\mu)$ is bounded with norm comparable to $\|\mu\|_\alpha$. Analogously, we say that $\mu$ is a *compact* $\alpha-$*Carleson measure* if the ratio in (1) goes to 0 uniformly in $\zeta$ as $\delta \to 0$, and it can be shown that this is equivalent to the map $I_\alpha$ being compact (see [MS], Theorem 4.3).

To apply these results to the weighted composition operator $wC_\phi : A^2_\alpha \to A^2_\beta$, where $\phi$ is an analytic self map of $D$ and $w$ is a bounded analytic function on $D$, employ a formal change of variable (see [H], pg. 163, Theorem C) to write

$$\begin{aligned} \|wC_\phi f\|^2_\beta &= \int_D |w(z)|^2 |f(\phi(z))|^2 d\lambda_\beta(z) \\ &= \int_D |f(z)|^2 d(|w|^2\lambda_\beta\phi^{-1})(z) \\ &= \|f\|^2_{L^2(|w|^2\lambda_\beta\phi^{-1})}, \end{aligned}$$

where $|w|^2\lambda_\beta\phi^{-1}$ is the measure on the open disk $D$ that assigns to each Borel set $F$ the measure

$$|w|^2\lambda_\beta\phi^{-1}(F) := \int_{\phi^{-1}(F)} |w(z)|^2 d\lambda_\beta(z) \tag{2}$$

It follows that $wC_\phi : A^2_\alpha \to A^2_\beta$ will be bounded (respectively compact) if and only if $|w|^2\lambda_\beta\phi^{-1}$ is an $\alpha-$Carleson (compact $\alpha$-Carleson) measure (cf. [MS], Corollary 4.4).

In the case where $wC_\phi$ acts from a weighted Bergman space to the Hardy space, we define a measure $|w|^2\sigma\phi^{-1}$ on the closed disk $\overline{D}$ that assigns to each Borel set $F$ the measure

$$|w|^2\sigma\phi^{-1}(F) := \int_{\phi^{*-1}(F)} |w^*(\zeta)|^2 d\sigma(\zeta) \tag{3}$$

where $\phi^*$ and $w^*$ denote the radial limit functions of $\phi$ and $w$. If $\phi$ has radial limits of modulus strictly less than 1 almost everywhere, this measure is actually supported on the open unit disk. Supposing this to be the case, the same comments as above show that the map $I_\alpha : A^2_\alpha \to L^2(|w|^2\sigma\phi^{-1})$ is bounded (compact) if and only if $|w|^2\sigma\phi^{-1}$ is an $\alpha-$Carleson (compact $\alpha-$Carleson) measure. Assuming for the moment that $p$ is a polynomial in $A^2_\alpha$, a formal change of variable yields

$$\begin{aligned}\|wC_\phi p\|^2_{H^2} &= \lim_{r\to 1}\int_{\partial D}|w(r\zeta)|^2\cdot|p(\phi(r\zeta))|^2 d\sigma(\zeta)\\ &= \int_{\partial D}|w^*(\zeta)|^2\cdot|p(\phi^*(\zeta))|^2 d\sigma(\zeta)\\ &= \int_D |p(z)|^2 d(|w|^2\sigma\phi^{-1})(z)\\ &= \|p\|^2_{L^2(|w|^2\sigma\phi^{-1})}\end{aligned}$$

where the second equality is justified by the fact that the integrand belongs to $H^2$ (see [D].) Since the polynomials are dense in $A^2_\alpha$, and for each polynomial $\|wC_\phi p\|_{H^2} = \|p\|_{L^2(|w|^2\sigma\phi^{-1})}$, it follows that for $\phi$ with radial limits of modulus strictly less than one almost everywhere, $wC_\phi : A^2_\alpha \to H^2$ is bounded (compact) if and only if $|w|^2\sigma\phi^{-1}$ is an $\alpha-$Carleson (compact $\alpha-$Carleson) measure.

Note that if $\phi$ has radial limits of modulus one on a set of positive measure, then $wC_\phi : A^2_\alpha \to H^2$ can be bounded only if $w$ is uniformly zero. This can be shown using the fact that each weighted Bergman space contains functions $f$ such that for each curve $\Gamma$ in $D$ tending to $\partial D$, $f$ assumes arbitrarily large values at points of $\Gamma$ (see [BES]). It is easy to see that if $f$ is such a function and for some $\zeta\in\partial D$ both $|\phi^*(\zeta)| = 1$ and $w^*(\zeta)\neq 0$, then $wf\circ\phi$ cannot have a radial limit at $\zeta$. In particular, if $\phi$ has radial limits of modulus 1 on a set $E$ of positive measure, then either $w\equiv 0$ or $w^*\neq 0$ almost everywhere on $E$; in the latter case, $wf\circ\phi$ has no radial limits on a set of positive measure, i.e. $wf\circ\phi\notin H^2$.

We summarize this discussion in the following proposition. The case where $wC_\phi$ acts from $A^2_\alpha$ to itself is included as a special case of $(a)$; we include $(c)$ to cover the case $wC_\phi : H^2\to H^2$. Although here we have not explicitly touched on $(c)$, it follows directly from the proof of Theorem 3.12 in [CM].

**Proposition 2.** *Let $\phi : D\to D$ be analytic, and suppose $w$ is a bounded analytic function, not identically zero on $D$. Let the measures $|w|^2\lambda_\beta\phi^{-1}$ and $|w|^2\sigma\phi^{-1}$ be as in* (2) *and* (3)*, respectively. Then:*

*$(a)$ $wC_\phi : A^2_\alpha\to A^2_\beta$ is bounded if and only if*

$$\left\||w|^2\lambda_\beta\phi^{-1}\right\|_\alpha := \sup_{S(\zeta,\delta)}\frac{|w|^2\lambda_\beta\phi^{-1}(S(\zeta,\delta))}{\delta^{\alpha+2}} < \infty$$

*$(b)$ $wC_\phi : A^2_\alpha\to H^2$ is bounded if and only $\phi$ has radial limits of modulus strictly less than 1 almost everywhere and*

$$\left\||w|^2\sigma\phi^{-1}\right\|_\alpha := \sup_{S(\zeta,\delta)}\frac{|w|^2\sigma\phi^{-1}(S(\zeta,\delta))}{\delta^{\alpha+2}} < \infty$$

*$(c)$ $wC_\phi : H^2\to H^2$ is bounded if and only if*

$$\left\||w|^2\sigma\phi^{-1}\right\|_{H^2} := \sup_{S(\zeta,\delta)}\frac{|w|^2\sigma\phi^{-1}(S(\zeta,\delta))}{\delta} < \infty.$$

*In all cases, the supremum is comparable to the norm of $wC_\phi$ acting on the appropriate spaces, and if the displayed quotient goes to 0 uniformly in $\zeta$ as $\delta\to 0$, then $wC_\phi$ is compact on these spaces.*

**Example:** Consider $\phi(z) = (1+z)/2$, and $\psi(z) = \phi(z) + t(z-1)^b$, where $b \geq 2$ and $t$ is small, as in [CM] pg. 337. (In particular, we may take $t < 1/128$.) It is known that on both the Hardy space and weighted Bergman spaces, $C_\phi - C_\psi$ is compact for $b \geq 2.5$ and not compact for $b = 2$. (See [CM], Ex. 9.3.3 for the Hardy space case, from which the Bergman case follows readily.) We use Theorem 1 and the Carleson conditions of Proposition 2 to resolve the intermediate cases.

We begin with the Bergman cases. By Theorem 1, $C_\phi - C_\psi : A^2_\alpha \to A^2_\alpha$ will be compact if the weighted composition operators $wC_{\phi_s} : A^2_{\alpha+2} \to A^2_\alpha$ are compact and uniformly norm bounded in $s$ for $s \in [0,1]$, where

$$w(z) = t(z-1)^b \tag{4}$$

and

$$\phi_s(z) = \frac{(1+z)}{2} + t(1-s)(z-1)^b \tag{5}$$

We will verify compactness and uniform boundedness by explictly computing the Carleson quotient of Proposition 2(a).

In what follows, $M$ represents a constant whose value may change from line to line, but which is always independent of $\zeta, \delta$ and $s$. To estimate the integral defining $|w|^2\lambda_\alpha\phi_s^{-1}(S(\zeta,\delta))$, note that if $z \in \phi_s^{-1}(S(\zeta,\delta))$, then $1 - |\phi_s(z)| \leq \delta$, and simple calculations show that

$$1 - |\phi_s(z)| \geq \frac{1}{16}|z-1|^2.$$

A little algebra now reveals that on $\phi_s^{-1}(S(\zeta,\delta))$,

$$|w(z)|^2 = t^2|z-1|^{2b} \leq t^2(16(1-|\phi_s(z)|))^b \leq M\delta^b. \tag{6}$$

Moreover, setting $w = 1$ and $\alpha = \beta$ in Proposition 2(a) shows that

$$\int_{\phi_s^{-1}(S(\zeta,\delta))} d\lambda_\alpha(z) \leq M\delta^{\alpha+2}\|C_{\phi_s}\|_\alpha, \tag{7}$$

where $\|C_{\phi_s}\|_\alpha$ denotes the norm of $C_{\phi_s}$ acting from $A^2_\alpha$ to itself and $M$ is a constant depending only on $\alpha$. Now it is known that for any analytic self map $\phi$ of the disk, the norm $\|C_\phi\|_\alpha$ is less than some multiple of a power of $\frac{1}{1-|\phi(0)|}$, where the multiple and the power depend only on $\alpha$. Since for all $s$, $|\phi_s(0)| \leq 1/2 + t$ which is bounded uniformly away from 1, the norms $\|C_{\phi_s}\|_\alpha$ are uniformly bounded in $s$.

Putting everything together we have, for fixed $\delta$,

$$\begin{aligned}
\sup_\zeta \frac{|w|^2\lambda_\alpha\phi_s^{-1}(S(\zeta,\delta))}{\delta^{\alpha+4}} &= \sup_\zeta \frac{\int_{\phi_s^{-1}(S(\zeta,\delta))}|w(z)|^2 d\lambda_\alpha(z)}{\delta^{\alpha+4}} \\
&\leq \frac{M\delta^b}{\delta^{\alpha+4}} \int_{\phi_s^{-1}(S(\zeta,\delta))} d\lambda_\alpha(z) && \text{(by 6)} \\
&\leq M\delta^{b-2} && \text{(by 7)} \qquad (8)
\end{aligned}$$

where the constant $M$ in the last line depends only on $\alpha$. We conclude that the operators $wC_{\phi_s} : A^2_{\alpha+2} \to A^2_\alpha$ are uniformly bounded in $s$ for $b \geq 2$, and compact for $b > 2$. It follows that $C_\phi - C_\psi : A^2_\alpha \to A^2_\alpha$ is compact for $b > 2$.

For the Hardy space, note that $\phi_s$ has radial limits of modulus strictly less than one almost everywhere, whence Proposition 2(b) shows that $wC_{\phi_s} : A^2_1 \to H^2$ will be bounded or compact as $|w|^2\sigma\phi_s^{-1}$ is a bounded or compact 1−Carleson measure.

Now estimate $|w^*(z)|$ for $z \in \phi_s^{*-1}(S(\zeta,\delta))$ by exactly the same technique that led to (6), and use Proposition 2(c) to show, as in (7), that

$$\int_{\phi_s^{*-1}(S(\zeta,\delta))} d\sigma(\zeta) \leq M\delta \|C_{\phi_s}\|_{H^2},$$

where $\|C_{\phi_s}\|_{H^2}$ denotes the norm of $C_\phi$ acting from $H^2$ to itself. Finally, invoke the uniform boundedness of these norms just as above to extract the same conclusion as in (8):

$$\sup_\zeta \frac{|w|^2 \sigma \phi_s^{-1}(S(\zeta,\delta))}{\delta^3} \leq M\delta^{b-2} \tag{9}$$

It follows that the $wC_{\phi_s} : A_1^2 \to H^2$ are uniformly bounded in $s$ for $b \geq 2$ and compact for $b > 2$. We conclude that $C_\phi - C_\psi : H^2 \to H^2$ is compact for $b > 2$.

We now use the proof Theorem 1 to obtain a sufficient condition for two composition operators to be path connected.

**Corollary 1.** *Let $\phi$ and $\psi$ be analytic self maps of the disk, and define $\phi_s(z) := s\phi(z) + (1-s)\psi(z)$ for $0 \leq s \leq 1$. Let $w(z)$ denote the bounded analytic function $\phi(z) - \psi(z)$. If the operators $wC_{\phi_s} : A_{\alpha+2}^2 \to A_\alpha^2$ are uniformly norm bounded in $s$, then $C_{\phi_s}$ is an arc of composition operators in $\mathcal{C}(A_\alpha^2)$. The same conclusion holds in $\mathcal{C}(H^2)$, provided the operators $wC_{\phi_s} : A_1^2 \to H^2$ are uniformly norm bounded.*

*Proof.* This follows from the calculations of Theorem 1 and the observation that

$$(C_{\phi_s} - C_{\phi_t})f(z) = (\phi(z) - \psi(z)) \int_s^t f'(\phi_r(z))dr.$$

In particular, if $f$ is a unit vector in $A_\alpha^2$, then

$$\begin{aligned}
\|(C_{\phi_s} - C_{\phi_t})f\|_{A_\alpha^2} &= \int_D |(C_{\phi_s} - C_{\phi_t})f(z)|^2 d\lambda_\alpha(z) \\
&= \int_D \left|(\phi(z) - \psi(z)) \int_s^t f'(\phi_r(z))dr\right|^2 d\lambda_\alpha(z) \\
&\leq \int_D |t-s| \int_s^t |w(z)f'(\phi_r(z))|^2 dr d\lambda_\alpha(z) \\
&= |t-s| \int_s^t \|wC_{\phi_r} f'\|_{A_\alpha^2}^2 dr.
\end{aligned}$$

By hypothesis, the integrand in the last line is bounded independently of $f$, by which the conclusion follows for the weighted Bergman spaces. For the Hardy space, start with a unit vector $f \in H^2$, replace the integral over $D$ by the integral that defines the Hardy space norm, and proceed identically. □

**Corollary 2.** *On both the Hardy space and any of the standard weighted Bergman spaces, there exist composition operators whose difference is not compact, yet which are arc-connected.*

*Proof.* Consider the $\phi$ and $\psi$ of the Example. As noted, $C_\phi - C_\psi$ is not compact for $b = 2$. However, the Carleson measure calculations in this example (in particular, lines (8) and (9)) show that for $b = 2$, the hypotheses of Corollary 1 are satisfied. The result follows. □

**Remark:** It is shown in [MOZ] that the maps of Corollary 2 induce an arc in $C(H^\infty)$ under the supremum norm. Using this fact, it is possible to obtain an alternative proof to the Hardy space case of Corollary 2 via an interpolation argument, since $(H^1, H^\infty)_\theta = H^2$ for $\theta = 1/2$ in the Calderon method of complex interpolation.

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