

USING THE EPQ WITH PARTIAL BACKORDERING FOR COORDINATED PLANNING OF A PRODUCT AND ITS COMPONENTS

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ABSTRACT

While there has been considerable work over the years on multistage lot-sizing models, particularly in an MRP environment, there has been relatively little work on systems recognizing the WIP effects when there is gradual, rather than batch, conversion of the components into the final product. We consider lot-sizing planning for a two-stage system in which the final product is planned using an EPQ model with partial backordering and the production of the components is controlled using basic EPQ models without backordering.

1. INTRODUCTION

The first work on the coordinated planning of a final product and its components that recognized the gradual conversion of input to output at a finite rate within each production stage was in papers by Banerjee and Burton (1989) and Banerjee et al. (1990) that assumed that the final product is planned with an EPQ model with no backordering and the components are ordered using basic EOQ models. Sharma and Singh (2004) extended this model to include using the EPQ, rather than the EOQ, model to plan component production and allowed for less-than-perfect production of the final product.

In this paper we extend this previous work by using Pentico et al.'s (2009) model for the single-product EPQ with partial backordering model as the starting point for a model for the coordinated planning of a two-level system consisting of a final product, which has the characteristics of an EPQ with partial backordering, and its immediate components, which are subject to control by an EPQ model without backordering.

2. THE BASIC MODEL

For the final product, we make all the usual assumptions of a deterministic EPQ model with full backordering except we assume that only a constant fraction β of the demands that occur during the stockout period will be backordered. In addition, we make the following assumptions:

- FIFO backorder filling for the final product, as discussed in Pentico et al. (2009), meaning that the oldest backorders are filled first;
- a one-stage final production system, as in Banerjee et al. (1990) and Sharma and Singh (2004);

- the final product has m components (without loss of generality, we assume that there is one unit of each component in the final product as in Banerjee et al. (1990) by redefining a component “unit” to consist of multiple original units, adjusting the holding cost and production rate for the component to compensate for the redefinition);
- perfect quality of the components;
- the production of components is according to an EPQ process with no backordering.

Parameters

D = demand per year for the final product

P = production rate per year of the final product if constantly producing

P_i = production rate per year of component i if constantly producing (Note: $P_i > P$ for feasibility of the model)

s = the unit selling price for the final product

C_o = the fixed cost of placing and receiving an order for the final product

C_{oi} = the fixed cost of placing and receiving an order for component i

C_p = the variable cost of producing a unit of the final product

C_{pi} = the variable cost of producing a unit of component i (included in the final product’s cost C_p)

C_h = the cost to hold a unit of the final product in inventory for a year

C_{hi} = the cost to hold a unit of component i in inventory for a year

C_b = the cost to keep a unit of the final product backordered for a year

C_g = the goodwill loss on a unit of unfilled demand for the final product

$C_l = (s - C_p) + C_g$ = the cost of a lost sale for the final product, including the lost profit on that unit and any goodwill loss

β = the fraction of stockouts of the final product that will be backordered

Variables

Q = the order quantity for the final product

Q_i = the order quantity for component i

T = the length of an order cycle for the final product

I = the maximum inventory level of the final product, with \bar{I} being the average inventory level over the year

\bar{I}_i = the average inventory level over the year for component i

S = the maximum stockout level of the final product, including both backorders and lost sales

B = the maximum backorder position for the final product, with \bar{B} being the average backorder level over the year ($B = \beta S$)

F = the fill rate or the percentage of demand for the final product that will be filled from stock

N_i = the number of times component i is produced/ordered during a production interval for the final product

The objective function combines the average cost per year for the final product, as shown in Pentico et al. (2009), with the average cost per year for the components:

$$\Gamma(T, F, N_1, \dots, N_m) = \frac{C_o + \sum_{i=1}^m N_i C_{oi}}{T} + C'_h \frac{DTF^2}{2} + \frac{TD^2[\beta + (1-\beta)F]^2}{2P} \sum_{i=1}^m \frac{C'_{hi}}{N_i} + C'_b \frac{\beta DT(1-F)^2}{2} + C_l D(1-\beta)(1-F) \quad (1)$$

where $C'_h = C_h(1-D/P)$, $C'_b = C_b(1-\beta D/P)$, and $C'_{hi} = C_{hi}(1-P/P_i)$.

3. ANALYTIC MODEL

While the cost function given in (1) should really be optimized over $T \in (0, \infty)$, $F \in [0, 1]$, and N_i positive integers, this is a complex, mixed-integer task that is computationally intensive and algorithmically tangled. In this section, therefore, we consider a relaxed version of this problem wherein we consider Γ as a function whose arguments can take any real value. The main result of this section is that this relaxed problem has a simple, closed-form analytic solution. In the remainder of this section, the symbol Γ will always represent the relaxed cost function.

Lemma 1. For fixed T and F , there is a unique set of N_i that minimizes Γ .

Setting the partials of Γ with respect to N_i equal to zero for each i yields a unique solution

$$N_i^* = \frac{DT[\beta + (1-\beta)F]}{\sqrt{2P}} \sqrt{\frac{C'_{hi}}{C_{oi}}}. \quad (2)$$

For fixed T and F , let N_1^*, \dots, N_m^* be as in (2), and define the function Γ^* as

$$\Gamma^*(T, F) = \Gamma(T, F, N_1^*, \dots, N_m^*).$$

Lemma 2. The function Γ^* has the form

$$\Gamma^*(T, F) = \frac{G_0}{T} + TR(F) + Q(F) \quad (3)$$

where $R(F)$ and $Q(F)$ are quadratic and linear polynomials, respectively, given by

$$R(F) := G_1 F^2 - 2G_2 F + G_2 \text{ and } Q(F) := G_3 F + G_4$$

with the coefficients G_i given by

$$G_0 = C_o \quad G_1 = \frac{D(C'_h + \beta C'_b)}{2} \quad G_2 = \frac{\beta C'_b D}{2} \quad (4)$$

$$G_3 = \frac{2D(1-\beta)}{\sqrt{2P}} \sum_{i=1}^m \sqrt{C'_{hi} C_{oi}} - C_l D(1-\beta) \quad G_4 = \frac{2D\beta}{\sqrt{2P}} \sum_{i=1}^m \sqrt{C'_{hi} C_{oi}} - C_l D(1-\beta)$$

Lemma 3. The cost function Γ has a unique minimizer.

The proof of this lemma does not specify where the minimizer lies. In particular, it could lie on the boundary, which is the subject of two corollaries.

Corollary 1. The value of F that minimizes Γ is 0 if and only if

$$G_3 - 2\sqrt{G_0 G_2} > 0. \quad (5)$$

Corollary 2. If (5) does not hold, then the minimizer lies on the boundary $F = 1$ if and only if

$$G_3 + 2\sqrt{G_0(G_1 - G_2)} < 0. \quad (6)$$

Otherwise, the minimizer is an interior point.

The above conditions relate to the position of the minimizer. The question remains of how to find it. One of the advantages of the form of the cost function in Lemma 2 is that it allows us to write down an explicit solution. This is the content of the following:

Lemma 4. Let (T^*, F^*) denote the (unique) pair that minimizes (3). Then F^* is the root of a quadratic polynomial

$$P(F) = aF^2 + bF + c \quad (7)$$

with coefficients

$$a = G_1G_3^2 - 4G_0G_1^2 \quad b = 8G_0G_1G_2 - 2G_2G_3^2 \quad c = G_2G_3^2 - 4G_0G_2^2 \quad (8)$$

and T^* satisfies

$$T^* = \frac{-G_3}{2(G_1F - G_2)}. \quad (9)$$

The minimizer F^* should be chosen as the root of (7) that makes T^* positive.

Using the results in the lemmas and corollaries, we define the solution procedure as follows:

Step 1: Use the equations in (4) to find the values of G_0 , G_1 , G_2 , and G_3 .

Step 2: From Corollary 1, if $G_3 - 2\sqrt{G_0G_2} > 0$, set $T = \infty$ and $F = 0$. The product should not be produced. If not, go to Step 3.

Step 3: From Corollary 2, if $G_3 + 2\sqrt{G_0(G_1 - G_2)} < 0$, set $F = 1$ and $T = \sqrt{2C_o / (DC'_h)}$. The product should be produced with no backordering. If not, go to Step 4.

Step 4: Use the equations in (8) to compute the values of a , b , and c for (7).

Step 5: Use the quadratic formula to determine two values of F from (7).

Step 6: Use F_1 and F_2 to determine two values of T from:

$$T = -G_3 / 2(G_1F - G_2)$$

Set $T^* = T_1$ or T_2 , whichever is positive. Set $F^* =$ the value of F that gave T^* .

Step 7: Use (2) m times to determine the values of N_1, N_2, \dots, N_m .

Example – Part 1: Finding the Continuous Optimum Solution

The product has four components. The parameter values are:

Final product: $C_o = 475$, $C_h = 95$, $C_b = 118.75$, $C_l = 296.875$, $D = 10$, $P = 120$, $\beta = 0.75$

Component 1: $C_{o1} = 0.25$, $C_{h1} = 20$, $P_1 = 480$

Component 2: $C_{o2} = 2.0$, $C_{h2} = 18$, $P_2 = 300$

Component 3: $C_{o3} = 0.15$, $C_{h3} = 17$, $P_3 = 180$

Component 4: $C_{o4} = 0.20$, $C_{h4} = 21$, $P_4 = 600$

Using the definitions given in Section 2, we compute the values of C'_h , C'_b , and C'_{hi} :

$$C'_h = 87.0833 \quad C'_b = 111.3281$$

$$C'_{h1} = 15 \quad C'_{h2} = 10.8 \quad C'_{h3} = 5.667 \quad C'_{h4} = 16.8$$

Step 1: Compute the values of G_0 , G_1 , G_2 , and G_3 from (4).

$$G_0 = 475 \quad G_1 = 852.90 \quad G_2 = 417.4805 \quad G_3 = -739.1733 \quad G_4 = 751.23$$

Step 2: Since $G_3 - 2\sqrt{G_0 G_2} = -739.1733 - 890.625 = -1629.8$ is not > 0 , go to Step 3.

Step 3: Since $G_3 + 2\sqrt{G_0(G_1 - G_2)} = -739.1733 + 909.5557 = 170.4$ is not < 0 , go to Step 4.

Step 4: Compute the values of a , b , and c from (8):

$$a = -916,120,132.9 \quad b = 896,854,372.3 \quad c = -103,049,073.4$$

Step 5: Use the quadratic formula to determine two values of F :

$$F_1, F_2 = \frac{-896,854,372.3 \pm \sqrt{(896,854,372.3)^2 - 4(-916,120,132.9)(-103,049,073.4)}}{2(-916,120,132.9)}$$

$$= 0.8460, 0.1330$$

Step 6: Use F_1 and F_2 to determine two values of T :

$$T_1 = 739.1733/2((852.90)(0.8460) - 417.4805) = 1.2154$$

$$T_2 = 739.1733/2((852.90)(0.1330) - 417.4805) = -1.2154$$

Since T_1 is positive, $T^* = T_1 = 1.2154$ and $F^* = F_1 = 0.8460$.

Step 7: Determine the values of the N_i^* s:

$$N_1^* = \frac{(10)(1.2154)[0.75 + (0.25)(0.8460)]}{\sqrt{2(120)}} \sqrt{\frac{15.00}{0.25}} = 5.8432$$

Similarly, $N_2^* = 1.7529$, $N_3^* = 4.6365$, and $N_4^* = 6.9137$.

As noted in both Banerjee et al. (1990) and Sharma and Singh (2004), the values of the N_i s should be integers, although the values determined from (2) will almost certainly not be. The minimum value of (1) would then be a lower bound on the optimal cost.

While it is possible, as noted by both Banerjee et al. (1990) and Sharma and Singh (2004), to solve for the optimal values of T , F , and the N_i s by using non-linear mixed integer programming, it is also possible, and much easier, to get very good solutions with integer values for the N_i s by using the integerization heuristic described in Banerjee et al. This procedure works as follows:

Step 1: As shown in Part 1 of the Example, find values for T^* and F^* , Substitute these values repeatedly into (2) to get the (probably non-integer) values for the N_i s. If all the N_i s are integers, the optimal solution has been found.

Step 2: Fix any N_i s with integer values at those values. Beginning with the non-integer N_i with the lowest index, let \underline{N}_i be the largest integer less than N_i and let \overline{N}_i be the smallest integer greater than N_i . (I.e., \underline{N}_i and \overline{N}_i "surround" N_i in Banerjee et al.'s terminology.)

Successively substitute \underline{N}_i and \overline{N}_i , along with any N_i s already fixed at integer values and the remaining non-integer N_i s into the objective function, (1). Fix the value of N_i for this index i to be the surrounding value that gives the lower objective function value. Repeat Step 2 until all components have had their value of N_i fixed as an integer. (Note: If any N_i is less than 1, its integerized value must be 1.)

Example – Part 2: Integerizing the N_i s from Part 1 of the Example

In Part 1 of the Example we determined that the optimal non-integer values of T , F , and the N_i s are: $T^* = 1.2154$, $F^* = 0.8460$, $N_1 = 5.8432$, $N_2 = 1.7529$, $N_3 = 4.6365$, and $N_4 = 6.9137$. All the

N_i s are non-integer. The value of the objective function is 907.50. The successive repetitions of Step 2 are as follows:

N_1 : The surrounding values for 5.8432 are 5 and 6. The cost with $N_1 = 6$ (907.50) is lower than the cost with $N_1 = 5$ (907.53), so fix $N_1 = 6$.

N_2 : The surrounding values for 1.7529 are 1 and 2. The cost with $N_1 = 6$ and $N_2 = 2$ (907.55) is lower than the cost with $N_1 = 6$ and $N_2 = 1$ (908.44), so fix $N_2 = 2$.

N_3 : The surrounding values for 4.6365 are 4 and 5. The cost with $N_1 = 6$, $N_2 = 2$, and $N_3 = 5$ (907.56) is lower than the cost with $N_1 = 6$, $N_2 = 2$, and $N_3 = 4$ (907.57), so fix $N_3 = 5$.

N_4 : The surrounding values for 6.9137 are 6 and 7. The cost with $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 7$ (907.56) is lower than the cost with $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 6$ (907.58), so fix $N_4 = 6$.

The “optimal” solution is then: $T = 1.2154$, $F = 0.8460$, $N_1 = 6$, $N_2 = 2$, $N_3 = 5$, and $N_4 = 7$, with a cost of 907.50, which is essentially the same as the lower bound found in Part 1 of the example.

4. COMPUTATIONAL STUDY

To gain some insights into how well Banerjee et al.’s (1990) integerization procedure works for this problem we identified six factors that we thought might make a difference in performance: number of components = 2 or 8; ratio of C_o to C_h for the final product = 0.5 or 5.0; ratio of C_l to C_b for the final product = 1.25 or 2.50; ratio of P to D for the final product = 3 or 10; value of β = 0.75 or 0.95; and the mix of desired ordering frequencies for the components is all monthly, all weekly, or half monthly and half weekly. The integerization procedure gave a solution for which the cost was less than 1.00005 times the cost of the lower bound in all 96 cases.

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A copy of the full paper, including descriptions and performance characteristics of three heuristics, may be obtained from the second author.