INTERMODULAR DESCRIPTION SHEET: UMAP Unit 753

TITLE: Computed Tomography in Multivariable Calculus

AUTHOR: Yves Nievergelt

Dept. of Mathematics, MS 32
Eastern Washington University
526 5th Street
Cheney, WA 99004–2431
ynievergelt@ewu.edu

MATHEMATICAL FIELD: Multivariable calculus

APPLICATION FIELD: Medicine, imaging

TARGET AUDIENCE: Students in multivariable calculus

ABSTRACT: This Module seeks to

• strengthen students’ intuition in multidimensional geometry,

• consolidate students’ command of multivariable integrals,

• provide exercises at a level between mechanical and theoretical, and

• show applications of calculus that are important but rarely appear in calculus texts.

This Module provides exercises with line integrals and multivariable integrals for use in multivariable calculus courses, at a level intermediate between mechanical problems and abstract proofs. The exercises do not require any background on any applications.

PREREQUISITES: In general: the ability to ponder a problem that does not come with a canned recipe for its solution. In particular, from multivariable calculus: limits and partial derivatives of elementary (rational, trigonometric, or exponential) functions, integrals along lines in the plane and along lines or planes in space, changes and permutations of variables in multivariable integrals, dot products, and small linear systems.


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(800) 77-COMAP = (800) 772-6627, or (617) 862-7878; http://www.comap.com
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Yves Nievergelt
Dept. of Mathematics, MS 32
Eastern Washington University
526 5th Street
Cheney, WA 99004–2431
ynievergelt@ewu.edu

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The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

Paul J. Campbell
Solomon Garfunkel
Editor
Executive Director, COMAP
Reinforcement of the thesis that sooner or later one’s favorite abstract mathematics will find application in the real world can be found in our experience with the x-ray CT problem.

— Lawrence A. Shepp [1983, 3]

1. Introduction

Computed tomography earned two of its inventors—G.N. Hounsfield of EMI and Allan M. Cormak of Tufts University—the Nobel Prize in Medicine in 1979 [Gelfand and Gindikin 1990, 1]. Computed tomography computes a picture of the interior of an object, on the basis of measurements taken only on the object’s boundary, without direct access to the object’s interior. For instance, medical scanners for computed tomography (CT) and nuclear magnetic resonance (NMR) produce pictures of a patient’s brain from x-rays or electromagnetic radiations measured only outside the patient’s head; computations directly construct pictures of the brain [Smith et al. 1977, 1231]. Thus, no camera ever photographs the brain: The scanner does not “take” any picture, it computes pictures. For examples of such pictures and their uses in medicine, see Sochurek [1987]. For an interactive demonstration of the use of computed tomography in an industrial nondestructive testing, see Bossi and Kruse [1994].

2. Physical Model of CT Scanners

The following considerations outline a physical explanation of the relation between line integrals and measurements of x-rays in a CT scanner.

An x-ray consists of photons that travel along a straight line \( L_{r,a} \) at a distance \( r \) from the origin and perpendicular to a direction \( \vec{a} \). At each point \( \vec{x} \) at position \( s \) on the line \( L_{r,a} \), if \( N(s) \) photons strike an object at \( s \) with density (or, more accurately, coefficient of linear absorption) \( f(\vec{x}(s)) \) and length \( h \) along the line \( L_{r,a} \), then the object will absorb some of the photons and will allow only a smaller number \( N(s + h) < N(s) \) to traverse and emerge at \( s + h \). In relative terms, the fraction of photons absorbed equals “approximately” the product of the density \( f(\vec{x}(s)) \) and the length \( h \) of the object,

\[
\frac{N(s) - N(s + h)}{N(s)} \approx f(\vec{x}(s)) \cdot h,
\]

whence

\[
\frac{N(s) - N(s + h)}{h \cdot N(s)} \approx f(\vec{x}(s)).
\]
The “approximation,” denoted by \( \approx \), means that equality holds at the limit as \( h \) tends to zero, i.e.,

\[
\lim_{h \to 0} \frac{N(s + h) - N(s)}{h \cdot N(s)} = -f(\vec{x}(s)),
\]

\[
\frac{N'(s)}{N(s)} = -f(\vec{x}(s)).
\]

Integrating this relation, between the penetration point (“in”) and exit point (“out”) of the x-ray, gives

\[
\ln(N(\text{out})) - \ln(N(\text{in})) = \int_{L_{r,a}} -f(\vec{x}(s)) \, ds,
\]

\[
\ln \left( \frac{N(\text{in})}{N(\text{out})} \right) = (\mathcal{R}f)(r,a).
\]

Thus, if the CT scanner measures the number or intensity of the x-ray photons before \( N(\text{in}) \) and after \( N(\text{out}) \) their passage through the body, then the scanner produces the numbers \( \ln (N(\text{in})/N(\text{out})) \), which form the values \( (\mathcal{R}f)(r,a) \) of the Radon transform of the tissue density \( f \) along the lines of the x-rays. The scanner’s computer must then reconstruct the yet unknown values of the tissue density \( f(\vec{x}) \) at many points \( \vec{x} \) in the body. Of course, someone must program the computer to this effect, for instance, as outlined in the following sections.

3. The Algebraic Reconstruction Technique (ART)

This section introduces the problem of computed tomography at the level of linear algebra without prerequisites from calculus. The subsequent sections, which use calculus, do not depend on this section.

3.1 Reconstructions from Projections Are Already In Linear Algebra

Though rarely described in such terms, reconstructions of objects from projections appear in most courses in linear algebra. Indeed, the coordinates \( x_1 \) and \( x_2 \) of a point \((x_1, x_2)\) in the plane are the orthogonal projections of \((x_1, x_2)\) onto the axes, and “reconstructing” the point \((x_1, x_2)\) from its projections \( x_1 \) and \( x_2 \) amounts to solving the trivial linear system

\[
\begin{align*}
x_1 &= x_1, \\
x_2 &= x_2.
\end{align*}
\]

This observation extends to all linear systems.
Example 1. Solving the system

\[ 0.6x_1 + 0.8x_2 = 0.7 \]
\[ 0.8x_1 - 0.6x_2 = 0.5 \]

gives \( x_1 = 0.82 \) and \( x_2 = 0.26 \). Such a solution corresponds to “reconstructing” the point \((x_1, x_2)\) from its orthogonal projections 0.7 and 0.5 on the lines supporting the unit vectors \((0.6, 0.8)\) and \((0.8, -0.6)\) respectively, as shown in Figure 1. With \( \langle , \rangle \) denoting the dot product, we have the system

\[ 0.6x_1 + 0.8x_2 = \langle (0.6, 0.8), (x_1, x_2) \rangle, \]
\[ 0.8x_1 - 0.6x_2 = \langle (0.8, -0.6), (x_1, x_2) \rangle. \]

Similarly, solving any linear system \( A\vec{x} = \vec{b} \) amounts to “reconstructing” the point \( \vec{x} \) in space from its scaled orthogonal projections on the lines supporting the not necessarily orthonormal rows of the matrix \( A \).

Other mathematical and applied concepts have algebraic or geometric features similar to those just outlined for linear systems, and, because of a lack of better words, such similar concepts also bear the name of “projection.” The following subsections illustrate such “projections” with prototype models of computed tomography. In many mathematical and applied situations, however, vectors become functions and the dot product becomes an inner product defined by integrals, as illustrated in subsequent sections with another model of computed tomography.
Exercises

1. A yet unknown point \((w_1, w_2)\) has orthogonal projections 2.284 and 1.812 on the lines supporting the vectors \((0.936, 0.352)\) and \((-0.352, 0.936)\). Determine \(w_1\) and \(w_2\).

2. A yet unknown point \((v_1, v_2)\) has orthogonal projections 0.26 and 0.18 on the lines supporting vectors \((0.96, -0.28)\) and \((0.28, 0.96)\). Determine \(v_1\) and \(v_2\).

3. A yet unknown point \((z_1, z_2, z_3)\) has orthogonal projections 3.16, 0.88, and 1.80 on the lines supporting the vectors \((0.36, 0.48, 0.80)\), \((0.48, 0.64, -0.60)\), and \((0.80, -0.60, 0.00)\), respectively. Determine \(z_1\), \(z_2\), and \(z_3\).

4. A yet unknown point \((v_1, v_2, v_3)\) has orthogonal projections 4, 8, and 1 on the lines supporting the vectors \((2/3, -2/3, 1/3)\), \((1/3, 2/3, 2/3)\), and \((2/3, 1/3, -2/3)\), respectively. Determine \(v_1\), \(v_2\), and \(v_3\).

3.2 Planar ART

CT scanners send x-rays through a planar cross section of a patient. Measuring instruments in the scanner then compare the intensities of the x-rays before they penetrate into the patient and after they emerge from the patient. Because the various tissues inside the patient absorb some of the x-rays’ energy, the intensity \(I_{\text{out}}\) of an x-ray emerging from the patient is smaller than the intensity \(I_{\text{in}}\) of the same x-ray as it entered the patient: \(I_{\text{out}} < I_{\text{in}}\). The ratios \(I_{\text{out}}/I_{\text{in}}\) measured outside the patient’s body constitute the only data measured by the scanners. From such data, the scanners’ manufacturers must design and program an algorithm to compute a picture showing the internal tissues in the patient. Because the absorption of an x-ray depends on the total mass traversed by that x-ray, the scanners’ computers must solve a system of equations to compute the density of mass at many points in the patient’s body to produce a picture useful for medical diagnostic purposes.

Figure 2 illustrates the role of mathematics in computed tomography. In the triangle, the three discs represent three internal organs at known locations but with yet unknown masses \(X_1\), \(X_2\), and \(X_3\), while the straight lines represent x-rays. In general, organs lie in front of one another, so that every x-ray traverses more than one organ. For instance, the x-ray along \(L_{12}\) traverses the total mass \(X_1 + X_2\) of the organs with masses \(X_1\) and \(X_2\), which absorb an amount \(B_{12}\) of the intensity of the x-ray. Thus, we assume the model \(X_1 + X_2 = B_{12}\). Similarly, \(X_2\) and \(X_3\) absorb an amount \(B_{23}\) from the x-ray along \(L_{23}\), while \(X_3\) and \(X_1\) absorb an amount \(B_{31}\) from the x-ray along \(L_{31}\), whence arises the linear system

\[
\begin{align*}
X_1 + X_2 &= B_{12}, \\
X_2 + X_3 &= B_{23}, \\
X_1 + X_3 &= B_{31}.
\end{align*}
\]
Figure 2. The three discs represent three yet unknown masses $X_1$, $X_2$, and $X_3$. The lines $L_{12}$, $L_{23}$, and $L_{31}$ represent three x-rays passing through the unknown masses. The sums $X_1 + X_2$, $X_2 + X_3$, and $X_3 + X_1$ represent measurements of the absorptions of the x-rays by the masses. In such a prototype setting, the problem of computed tomography consists in designing a method to compute the three masses $X_1$, $X_2$, and $X_3$ from the measurements $X_1 + X_2$, $X_2 + X_3$, and $X_3 + X_1$.

Because the system has only a single solution, measuring the absorptions $(B_{12}, B_{23}, B_{31})$ and solving the resulting linear system (through elimination or with a computer) reveals the masses $(X_1, X_2, X_3)$.

Example 2. Suppose that the scanner produces the measurements $B_{12} = 0.8$, $B_{23} = 0.9$, and $B_{31} = 0.7$. The linear system becomes

\[
\begin{align*}
X_1 + X_2 &= 0.8 \\
X_2 + X_3 &= 0.9 \\
X_1 + X_3 &= 0.7.
\end{align*}
\]

Solving the system gives masses $X_1 = 0.3$, $X_2 = 0.5$, and $X_3 = 0.4$.

For medical diagnostics, computed tomography calculates the density of tissues not just at three locations but at thousands of locations in each organ. Hence arise linear systems of thousands of equations with thousands of unknowns, with the unknown $X_N$ denoting the density (or, more accurately, the “coefficient of linear absorption”) at the $N$th selected point (Herman [1979]).
Figure 3. The four discs represent four yet unknown masses $X_1$, $X_2$, $X_3$, and $X_4$. The lines represent six x-rays passing through the unknown masses. The sums denote measurements of the absorptions of the x-rays by the masses. In such a prototype setting, the problem of computed tomography consists in designing a method to compute the four masses $X_1$, $X_2$, $X_3$, and $X_4$ from the measurements.

Indeed, four points already cause problems, because they determine not four but six x-rays through them, as in Figure 3. Yet six equations with four unknowns may have no solution. Moreover, selecting only four among six x-rays may still give a singular system without solution. Furthermore, while exact formulas exist to calculate each unknown $X_N$ in terms of all the x-rays through all pairs of masses [Strichartz 1982], such formulas do not help in practice because the locations of the masses remain unknown.

Exercises

5. As in Example 2, and with the same situation as in Figure 2, suppose that the scanner produces the measurements $B_{12} = 0.9$, $B_{23} = 0.5$, and $B_{31} = 0.8$. Compute the masses $X_1$, $X_2$, and $X_3$.

6. As in Example 2, and with the same situation as in Figure 2, suppose that the scanner produces the measurements $B_{12} = 0.9$, $B_{23} = 0.4$, and $B_{31} = 0.9$. Compute the masses $X_1$, $X_2$, and $X_3$.

3.3 Spatial ART

In contrast to CT scanners, which measure x-rays along lines in the plane, nuclear magnetic resonance (NMR) scanners measure electromagnetic fields
over planes in space; but the mathematics is similar. For examples of pictures computed by NMR, also called magnetic resonance imaging (MRI) in medicine, see Sochurek [1987, 14–23].

Consider, for instance, four yet unknown masses \( X_1, X_2, X_3, \) and \( X_4 \) in space. In general position, they determine four planes through them. (Four noncoplanar masses lie at the vertices of a tetrahedron, which has four planar triangular faces, as shown in Figure 4.) With \( I \) labeling both the plane \( P_I \) and the measurement \( B_I \) that omits \( X_I \), the NMR measurements correspond to the sum of the three masses on each plane, which give four equations:

\[
\begin{align*}
X_2 + X_3 + X_4 &= B_1 \quad &\text{(sum along the plane } P_1 \text{ through } X_2, X_3, X_4), \\
X_1 + X_3 + X_4 &= B_2 \quad &\text{(sum along the plane } P_2 \text{ through } X_1, X_3, X_4), \\
X_1 + X_2 + X_4 &= B_3 \quad &\text{(sum along the plane } P_3 \text{ through } X_1, X_2, X_4), \\
X_1 + X_2 + X_3 &= B_4 \quad &\text{(sum along the plane } P_4 \text{ through } X_1, X_2, X_3).
\end{align*}
\]

**Example 3.** For the four measurements \((B_1, B_2, B_3, B_4) = (7, 6, 9, 8)\), the system becomes

\[
\begin{align*}
X_2 + X_3 + X_4 &= 7, \\
X_1 + X_3 + X_4 &= 6, \\
X_1 + X_2 + X_4 &= 9, \\
X_1 + X_2 + X_3 &= 8.
\end{align*}
\]

Hence, using elimination, a calculator, or a computer yields the solution \((X_1, X_2, X_3, X_4) = (3, 4, 1, 2)\).

The four masses serve only as a prototype to explain the nature of NMR scanners. Accuracy and resolution sufficient for medical diagnostic purposes require thousands of points, whence arise systems with thousands of equations.
For such and other reasons, the method just described—called the *algebraic reconstruction technique* (ART)—has remained confined to some particular applications [Natterer 1986, 138 and 160]. A more detailed presentation of ART for use in a linear algebra appears in Rorres [1984].

For general computed tomography, however, a different method, based on multivariable calculus and related mathematical topics, has proved more effective, as described in the following sections.

There are still other types of computed tomography, for instance, with x-rays not only within a plane but also through three-dimensional space [Quinto et al. 1994]. For pictures computed by such methods, see Sochurek [1987, 34–39].

**Exercises**

7. In the situation of Example 3, calculate the masses \((X_1, X_2, X_3, X_4)\) on the basis of the measurements \((B_1, B_2, B_3, B_4) = (9, 5, 8, 8)\).

8. In the situation of Example 3, calculate the masses \((X_1, X_2, X_3, X_4)\) on the basis of the measurements \((B_1, B_2, B_3, B_4) = (6, 5, 7, 6)\).

**4. Radon’s Transform**

Whereas the algebraic reconstruction technique models a patient’s internal tissues with finitely many points or cells, the method presented in this section models internal tissues with a continuous function of two variables that represents the density of tissue at each point of a planar cross section of the patient. The measurements from a CT scanner then depend on the total mass traversed by each x-ray; these measurements correspond to the integrals of the density over each straight line in the plane. With a large number of points, continuous models such as this one have proved more effective than discrete models. This section presents these ideas in detail.

**4.1 The Set of All Straight Lines in the Plane**

To assign specific measurements to specific x-rays, it becomes necessary to specify the position of each x-ray. With x-rays modeled by straight lines, however, specifying the position of each x-ray amounts to specifying the location and orientation of each straight line in the plane. A common method specifies each straight line \(L_{r,a} \subset \mathbb{R}^2\) in the plane \(\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}\) by two real numbers: the polar coordinates \(r\) and \(a\) of the point \(r \cdot (\cos a, \sin a)\) on \(L_{r,a}\) closest to the origin, usually—but not always—with the convention that \(r \in \mathbb{R}\) and \(0 \leq a < \pi\). Thus, \(r\) represents the signed distance measured perpendicularly from the line \(L_{r,a}\) to the origin \(\vec{0} = (0, 0)\), and \(a\) represents the angle measured counterclockwise from the positive horizontal axis to the direction perpendicular to \(L_{r,a}\), as in Figure 5.
Figure 5. For each line $L_{r,a}$ in the plane, $r$ and $a$ denote the polar coordinates of the point on $L_{r,a}$ closest to the origin.

Example 4. For the horizontal line at height 1 above the horizontal axis, $r = 1$ and $a = \pi/2$.

Also, for the vertical line at distance 1 on the left of the vertical axis, $r = -1$ and $a = 0$.

Moreover, for the line through the points $(1,0)$ and $(0,1)$, $r = \sqrt{2}/2$ and $a = \pi/4$.

Furthermore, for the line through $(-1,0)$ and $(0,-1)$, $r = -\sqrt{2}/2$ and $a = \pi/4$.

Figure 6. For each point $\vec{x} = (x,y)$ on the line $L_{r,a}$, if $\vec{a} = r \cdot (\cos a, \sin a)$ denotes the point on $L_{r,a}$ closest to the origin, then $\langle \vec{x}, \vec{a} \rangle = r$.

The specification of lines just presented, with $r \in \mathbb{R}$ and $0 \leq a < \pi$, has the advantage of allowing for all straight lines with all positions and orientations in the plane. In contrast, specifications involving slopes do not allow for vertical lines. Yet the two numbers $r$ and $a$ provide enough information
to recover the Cartesian equation of a line. Indeed, let \( r \cdot (\cos a, \sin a) \) represents the point closest to the origin on the line \( L_{r,a} \), as in Figure 5. Then the line \( L_{r,a} \) is perpendicular to the unit vector \( \vec{a} = (\cos a, \sin a) \). Consequently, the orthogonal projection of every point \( \vec{x} = (x, y) \) on \( L_{r,a} \) coincides with the point \( r \cdot (\cos a, \sin a) \). Indeed, the orthogonal projection of the entire line \( L_{r,a} \) reduces to the point \( r \cdot (\cos a, \sin a) \), as in Figure 6. Therefore, with \( \langle \cdot, \cdot \rangle \) denoting the dot product,

\[
\langle \vec{x}, \vec{a} \rangle = x \cos a + y \sin a = r
\]

is a Cartesian equation for the straight line \( L_{r,a} \).

**Example 5.** The line \( L_{r,a} \) specified by the polar coordinates \( r = 5 \) and \( a = \pi/3 \) satisfies the Cartesian equation

\[
\begin{align*}
x \cos a & + y \sin a = r, \\
x(1/2) & + y(\sqrt{3}/2) = 5.
\end{align*}
\]

The two numbers \( r \) and \( a \) also give a parametric equation for the straight line \( L_{r,a} \). The line \( L_{r,a} \) passes through \( r\vec{a} \), and \( L_{r,a} \) is parallel to the direction \( \vec{n}_{\vec{a}} = (-\sin a, \cos a) \) perpendicular to \( \vec{a} \). Consequently, for each point \( \vec{x} = (x, y) \) on \( L_{r,a} \), if \( s \) denotes the Euclidean distance from \( \vec{x} \) to \( r\vec{a} \), counted positively along \( \vec{n}_{\vec{a}} \) and negatively in the opposite direction \( -\vec{n}_{\vec{a}} \), then

\[
\vec{x} = r\vec{a} + s \cdot (\vec{n}_{\vec{a}})
\]

\[
= r \cdot (\cos a, \sin a) + s \cdot (-\sin a, \cos a)
\]

\[
= (r \cos a - s \sin a, r \sin a + s \cos a).
\]

This equation parametrizes the line \( L_{r,a} \) by its arclength, \( s \).
Example 6. The line $L_{r,a}$ specified by the polar coordinates $r = 5$ and $a = \pi/3$ has the parametric equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos a - s \sin a \\ r \sin a + s \cos a \end{pmatrix} = \begin{pmatrix} 5 \cdot (1/2) - s \cdot (\sqrt{3}/2) \\ 5 \cdot (\sqrt{3}/2) + s \cdot (1/2) \end{pmatrix}.$$ \hspace{1cm} (3)

Exercises

9. Determine a Cartesian equation for the straight line corresponding to the polar coordinates $r = -7$ and $a = 5\pi/6$.

10. Determine a Cartesian equation for the straight line corresponding to the polar coordinates $r = -5$ and $a = 3\pi/4$.

11. Determine a parametric equation for the straight line corresponding to the polar coordinates $r = -7$ and $a = 5\pi/6$.

12. Determine a parametric equation for the straight line corresponding to the polar coordinates $r = -5$ and $a = 3\pi/4$.

13. Use analytic geometry to obtain a formula for the point $r \cdot (\cos a, \sin a)$ closest to the origin and on the line $L_{r,a}$ with Cartesian equation $ux + vy = w$. Express $r$ and $a$ in terms of $u$, $v$, and $w$.

14. Use Lagrange multipliers to obtain a formula for the point $r \cdot (\cos a, \sin a)$ closest to the origin and on the line $L_{r,a}$ with Cartesian equation $ux + vy = w$. Express $r$ and $a$ in terms of $u$, $v$, and $w$.

4.2 Integrals of Functions over Straight Lines

This subsection explains the nature of the measurements from a CT scanner in terms of line integrals.

Let $\mathcal{D} \subset \mathbb{R}^2$ denote the subset of the plane corresponding to a patient’s cross section. For definiteness, assume that $\mathcal{D}$ is closed and bounded. Also, consider a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y)$ models the density (more accurately, the coefficient of linear absorption) of the internal tissues at the point $(x, y)$; in particular, $f(x, y) = 0$ for every point $(x, y)$ outside $\mathcal{D}$.

Since we use the notation $(p, q)$ to denote a point, let the notation $]p, q[$ represent the open segment from $p$ to $q$ (neither endpoint is included): $]p, q[ = \{x \in \mathbb{R} : p < x < q\}$. Similarly, $[p, q]$ and $]p, q]$ stand for half-open segments.

Definition 1. The Radon transform of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the function

$$(\mathcal{R}f) : \mathbb{R} \times [0, \pi] \to \mathbb{R}$$

defined by the integrals of $f$ along each straight line in the plane:

$$(\mathcal{R}f)(r, a) = \int_{L_{r,a}} f \, ds = \int_{-\infty}^{\infty} f(\sqrt{r \cos a - s \sin a}, \sqrt{r \sin a + s \cos a}) \, ds.$$ 

(3)

11
Formula (3) results from a substitution into $f(x, y)$ of the parametric equation of the line, with $x = r \cos a - s \sin a$ and $y = r \sin a + s \cos a$, as derived in (1).

**Figure 8.** The function $f_{R,c}$ with $f_{R,c}(x, y) = c \cdot \left(1 - \left[\sqrt{x^2 + y^2} / R\right]\right)$, where $x^2 + y^2 \leq R^2$.

**Example 7.** Consider the function $f_{R,c}$ shown in Figure 8, with value $f_{R,c} = c \left(1 - \left[\sqrt{x^2 + y^2} / R\right]\right)$ in the closed disc $D$ where $x^2 + y^2 \leq R^2$, and constant value 0 everywhere else:

$$f_{R,c}(x, y) = \begin{cases} c \left(1 - \left[\sqrt{x^2 + y^2} / R\right]\right), & \text{if } x^2 + y^2 \leq R^2; \\ 0, & \text{if } x^2 + y^2 > R^2. \end{cases}$$

For each straight line $L_{r,a}$ with polar coordinates $r$ and $a$, two cases arise. If $r > R$, then $L_{r,a}$ does not intersect disc $D$, so that $f_{R,c} = 0$ everywhere along $L_{r,a}$, and, consequently, $(Rf_{R,c})(r, a) = 0$.

**Figure 9.** $d = \sqrt{R^2 - r^2}$. 
However, if \( r \leq R \), then \( L_{r,a} \) intersects \( D \) along a line segment of length \( 2d \) given by \( r^2 + d^2 = R^2 \), whence \( d = \sqrt{R^2 - r^2} \), as shown in Figure 9. The segment extends to a distance \( d \) parallel to \( L_{r,a} \) on either side of the point \( r \mathbf{a} \) on \( L_{r,a} \), because \( D \) and \( L_{r,a} \) are each symmetric with respect to the diameter supporting \( \mathbf{a} \).

![Diagram](image)

**Figure 10.** The value \((\mathcal{R}f_{R,c})(r,a)\) of the Radon transform represents the integral (shaded area) of \( f_{R,c} \) over the line \( L_{r,a} \).

Hence, the value of the Radon transform, which is the integral over the shaded area in Figure 10, is

\[
(\mathcal{R}f_{R,c})(r,a) = \int_{-\infty}^{\infty} f_{R,c}(r \cos a - s \sin a, r \sin a + s \cos a) \, ds
\]

\[
= \int_{-d}^{d} c \left\{ 1 - \frac{\sqrt{(r \cos a - s \sin a)^2 + (r \sin a + s \cos a)^2}}{R} \right\} \, ds
\]

\[
= \int_{-\sqrt{R^2 - r^2}}^{\sqrt{R^2 - r^2}} c \left\{ 1 - \frac{\sqrt{r^2 + s^2}}{R} \right\} \, ds
\]

\[
= 2c \left\{ s - \frac{1}{R} \left[ \frac{s}{2} \sqrt{r^2 + s^2} + \frac{r^2}{2} \arcsinh \left( \frac{s}{r} \right) \right] \right\}_{0}^{\sqrt{R^2 - r^2}}
\]

\[
= 2c \left\{ \sqrt{R^2 - r^2} - \frac{1}{R} \left[ \frac{\sqrt{R^2 - r^2}}{2} R + \frac{r^2}{2} \arcsinh \left( \frac{\sqrt{R^2 - r^2}}{r} \right) \right] \right\}
\]

\[
= c \left\{ \sqrt{R^2 - r^2} - \frac{r^2}{R} \arcsinh \left( \frac{\sqrt{R^2 - r^2}}{r} \right) \right\}.
\]

In practice, each line \( L_{r,a} \) represents an x-ray, and the numbers \((\mathcal{R}f)(r,a)\) represent the measurements provided by the CT scanner. The problem of computed tomography then consists in starting from the measurements \((\mathcal{R}f)(r,a)\) and solving for the values \( f(x,y) \) of the density of the internal tissues. In
particular, while the theory of Radon transforms proves indispensable in providing guidance to design scanners, hardly anyone ever calculates the explicit functional form of the Radon transform of a specific function, except for a small number of test cases—for instance, as in Example 7 and in the following exercises. All practical calculation is done with numerical methods.

**Exercises**

The results of the following exercises are useful as test cases for the design of CT scanners. Such test cases often simulate cross sections of the skull and other internal organs by superpositions of ellipses with various densities [Shepp 1983].

15. Calculate the Radon transform \((R g_{R,c})(r,a)\) of the function \(g_{R,c}\) with constant value \(c\) in the closed disc \(D\) where \(x^2 + y^2 \leq R^2\), and constant value 0 everywhere else:

\[
g_{R,c}(x, y) = \begin{cases} 
  c, & \text{if } x^2 + y^2 \leq R^2; \\
  0, & \text{if } x^2 + y^2 > R^2. 
\end{cases}
\]

16. Calculate the Radon transform \((R h_{R,c})(r,a)\) of the function \(h_{R,c}\) whose graph represents one-half of an ellipsoid on the closed disc \(D\) where \(x^2 + y^2 \leq R^2\), with constant value 0 everywhere else:

\[
h_{R,c}(x, y) = \begin{cases} 
  c \sqrt{R^2 - (x^2 + y^2)}, & \text{if } x^2 + y^2 \leq R^2; \\
  0, & \text{if } x^2 + y^2 > R^2. 
\end{cases}
\]

17. Calculate the Radon transform \((R g_{p,q,c})(r,a)\) of the function \(g_{p,q,c}\) with constant value \(c\) in the closed ellipse \(E\) where \(\left(x^2/p^2\right) + \left(y^2/q^2\right) \leq 1\) and constant value 0 everywhere else:

\[
g_{p,q,c}(x, y) = \begin{cases} 
  c, & \text{if } x^2/p^2 + y^2/q^2 \leq 1; \\
  0, & \text{if } x^2/p^2 + y^2/q^2 > 1. 
\end{cases}
\]

Verify that if \(p = q\), then the result reduces to that for the disc in Exercise 15.

18. (This exercise demonstrates how, in practice, the challenge of changes of coordinates arises not from the mechanical execution but from setting up changes of coordinates.) Apply changes of coordinates to Exercise 17 to calculate the Radon transform \((R g_{u,v,w,p,q,c})(r,a)\) of the function \(g_{u,v,w,p,q,c}\) with constant value \(c\) inside the closed ellipse \(E\) with center at \((u,v)\), angle \(w\) from the horizontal axis to the semimajor axis, and semimajor and semiminor axes with positive lengths \(p\) and \(q\), and constant value 0 everywhere else:

\[
g_{u,v,w,p,q,c}(x, y) = \begin{cases} 
  c, & \text{if } (x,y) \text{ lies inside or on } E; \\
  0, & \text{if } (x,y) \text{ lies outside } E. 
\end{cases}
\]
Hint: express one set of coordinates as a combination of a rotation and a translation (shift) of the other set of coordinates; invert the transformation if the need arises.

19. Verify that if \( f \) does not depend on \( a \), then neither does \( Rf \).

20. Verify that for all numbers \( p, q \in \mathbb{R} \) and for all continuous functions \( f, h : \mathbb{R}^2 \to \mathbb{R} \) (which equal zero outside of \( D \)), \( R(pf + qh) = p \cdot Rf + q \cdot Rh \).

5. Approximate Identities

In practice, measurements may suffer from inaccuracies, for instance, caused by limitations in the measuring apparatus or unforeseen external disturbances. Nevertheless, such inaccuracies may cancel one another on the average, so that the average of many measurements may be more accurate than any single measurement. Therefore, practical computations of an image from x-ray measurements involve averages, usually weighted averages with larger weights near the location of interest and smaller weights farther away, as described in this section.

5.1 Weight Functions with One Dimension

The usual average of a continuous function \( f : [a, b] \to \mathbb{R} \) over a closed and bounded interval \([a, b]\) with \( a < b \) is

\[
\frac{1}{b - a} \int_{a}^{b} f(t) \, dt = \int_{a}^{b} f(t) \cdot \frac{1}{b - a} \, dt.
\]

Such an average “assigns” the same “weight” \( 1/(b - a) \) to each value \( t \in [a, b] \). Yet many situations require a larger weight near a particular location of interest and smaller weights farther away. Hence arises a more general and useful concept of weighted average.

**Definition 2.** A one-dimensional weight function is a function \( w : \mathbb{R} \to \mathbb{R} \) such that \( \int_{\mathbb{R}} |w(t)| \, dt < \infty \) and such that

\[
\int_{\mathbb{R}} w(t) \, dt = 1.
\]

Specific applications may impose further conditions, for instance, nonnegativity, continuity, or differentiability.

In the sequel, the notation \( A := B \) serves to define a yet undefined object \( A \) in terms of an already defined object \( B \).

**Example 8.** The function \( h : \mathbb{R} \to \mathbb{R} \) (with \( h \) for “hat”) shown in Figure 11a and defined by
Figure 11. a. A weight function $h$. b. $h$ shifted by $u$ to the right. c. $h$ compressed by the multiplicative factor $c$ horizontally and stretched by $1/c$ vertically. d. $h$ shifted by $u$, compressed by $c$ horizontally, and stretched by $1/c$ vertically.

$$h(t) := \begin{cases} 
0, & \text{if } t < -1; \\
1 + t, & \text{if } -1 \leq t < 0; \\
1 - t, & \text{if } 0 \leq t < 1; \\
0, & \text{if } 1 \leq t 
\end{cases}$$ (4)

is a weight function, because $\int_{\mathbb{R}} h(t) \, dt = 1$. The observation that $h$ is an even function, which means that $h(-t) = h(t)$, simplifies the verification:

$$\int_{\mathbb{R}} h(t) \, dt = \int_{-1}^{0} (1 + t) \, dt + \int_{0}^{1} (1 - t) \, dt = 2 \int_{0}^{1} (1 - t) \, dt$$
$$= 2 \left( t - \frac{t^2}{2} \right) \bigg|_{0}^{1} = 2 \left( 1 - \frac{1}{2} \right) = 1.$$

Moreover, $h$ is also nonnegative and continuous.

The weight function in Example 8 assigns the largest weight to the origin, $t = 0$, and no weight at all outside the interval $[-1, 1]$. Still, many situations require the possibility of adjusting the relative size of the largest weights and the interval where most of the weight lies.

**Example 9.** For each $c > 0$, the function $h_c : \mathbb{R} \rightarrow \mathbb{R}$ shown in Figure 11c and defined by
is a weight function, because $\int_{\mathbb{R}} h_c(t) \, dt = 1$:

$$
\int_{\mathbb{R}} h_c(t) \, dt = \int_{-c}^{0} \frac{1}{c} \cdot \left(1 + \frac{t}{c}\right) \, dt + \int_{0}^{c} \frac{1}{c} \cdot \left(1 - \frac{t}{c}\right) \, dt
= 2 \int_{0}^{c} \frac{1}{c} \left(1 - \frac{t}{c}\right) \, dt = 2 \frac{1}{c} \left(t - \frac{t^2}{2c}\right) \bigg|_{0}^{c} = 2 \frac{1}{c} \left(c - \frac{c^2}{2c}\right) = 1.
$$

Thus, $h_c$ is a weight function that assigns the largest weight to the origin, $t = 0$, with no weight outside $[-c, c]$. Moreover, $h_c$ is also nonnegative and continuous.

With any weight function $w$ that assigns most of its weight near the origin, composition of sign-reversal and the translation (shift) defined by $t \mapsto u - t$ produce a weight function $t \mapsto w(u - t)$ that assigns most of its weight near $u$.

**Example 10.** See Figure 11b. Consider the weight function $h$ defined by (4) in Example 8. For each $u \in \mathbb{R}$,

$$
h(u - t) = \begin{cases} 
0, & \text{if } u - t < -1, \text{ that is, if } u + 1 < t; \\
1 + (u - t), & \text{if } -1 \leq u - t < 0, \text{ that is, if } u < t \leq u + 1; \\
1 - (u - t), & \text{if } 0 \leq u - t < 1, \text{ that is, if } u - 1 < t \leq u; \\
0, & \text{if } 1 \leq u - t, \text{ that is, if } t \leq u - 1.
\end{cases}
$$

The shifted function remains a weight function, because the change of variable $s := u - t$ gives

$$
\int_{\mathbb{R}} |h(u - t)| \, dt = \int_{\mathbb{R}} h(u - t) \, dt = \int_{u-1}^{u+1} h(u - t) \, dt
= \int_{-1}^{1} h(s) (-ds) = \int_{-1}^{1} h(s) \, ds = 1.
$$

**Example 11.** For each $c > 0$, consider the function $h_c : \mathbb{R} \to \mathbb{R}$ defined in Example 9 by (5). For each $u \in \mathbb{R}$, the composition of the weight function $h_c$ and the change of variable $t \mapsto u - t$ produces the shifted weight function shown in Figure 11d, with
\[ h_c(u-t) = \begin{cases} 
0, & \text{if } u-t < -c, \text{ that is, if } u+c < t; \\
\frac{1}{c} \left( 1 + \frac{u-t}{c} \right), & \text{if } -c \leq u-t < 0, \text{ that is, if } u < t \leq u+c; \\
\frac{1}{c} \left( 1 - \frac{u-t}{c} \right), & \text{if } 0 \leq u-t < c, \text{ that is, if } u-c < t \leq u; \\
0, & \text{if } c \leq u-t, \text{ that is, if } t \leq u-c. 
\end{cases} \]

Exercises

21. Verify that for each \( c > 0 \), the function \( b_c : \mathbb{R} \rightarrow \mathbb{R} \) (with \( "b" \) for "box") defined by
\[
b_c(t) := \begin{cases} 
0, & \text{if } t < -c; \\
\frac{1}{2c}, & \text{if } -c \leq t \leq c; \\
0, & \text{if } c < t,
\end{cases}
\]
is a weight function.

22. (The function introduced here, known as Dirichlet’s kernel, plays an important role in the theory of Fourier series [Hewitt and Stromberg 1965, 291–292].) Verify that for each positive integer \( N \), the function \( D_N \) defined by
\[
D_N(t) := \frac{\sin \left( \left\lfloor N + \frac{1}{2} \right\rfloor t \right)}{2\pi \sin(t/2)}
\]
for \( t \neq 0 \) and \( D_N(0) := 2N + 1 \) is a weight function on \([-\pi, \pi]\), so that \( \int_{-\pi}^{\pi} D_N(t) \, dt = 1 \). Hints: by induction or otherwise, verify the trigonometric identity
\[
\left( \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos Nt \right) \cdot 2 \sin t = \sin \left( N + \frac{1}{2} \right)t;
\]
then use this identity to derive an algebraically equivalent formula for \( D_N \) that is longer but easier to integrate.

23. (The function introduced here, known as Abel’s kernel, plays an important role in the theory of Fourier transforms [Hewitt and Stromberg 1965, 407 and 409].) Verify that for each positive real \( B > 0 \), the function \( A_B : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
A_B(w) := \frac{1}{\pi} \cdot \frac{B}{B^2 + w^2}
\]
is a weight function on \( \mathbb{R} \), so that \( \int_{-\infty}^{\infty} A_B(w) \, dw = 1 \).

24. (The function introduced here, known as Fejér’s kernel, plays an important role in the theory of Fourier transforms [Hewitt and Stromberg 1965, 407–408 and 413].) Verify that for each positive real \( B > 0 \), the function \( F_B : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[ F_B(w) := \frac{B}{2\pi} \left[ \frac{\sin(Bw/2)}{Bw/2} \right]^2 \]

for \( w \neq 0 \) and \( F_B(0) := B/(2\pi) \) is a weight function on \( \mathbb{R} \), so that \( \int_{-\infty}^{\infty} F_B(w) \, dw = 1 \).

### 5.2 Approximate Identities with One Dimension

Any collection of such weight functions as \( h_c \) in Example 9 that allows for the adjustment of the interval containing most of the weight bears the name of “approximate identity,” for reasons explained below.

**Definition 3.** An approximate identity is a set of weight functions, \( \{w_c : c > 0\} \), such that the smaller the value of \( c \), the smaller the interval in which the weight function \( w_c \) assigns most of the weight: for each interval \( [-R,R] \) with \( R > 0 \),

\[
\lim_{c \to 0} \int_{-R}^{R} w_c(t) \, dt = 1,
\]

and such that some \( M > 0 \) exists for which \( \int_{\mathbb{R}} |w_c(t)| \, dt \leq M \) for every \( c > 0 \).

**Example 12.** The set \( \{h_c : c > 0\} \) defined by (5) in Example 9 forms an approximate identity. Indeed, for each \( R > 0 \), if \( 0 < c < R \), then

\[
\int_{-R}^{R} |h_c(t)| \, dt = \int_{-R}^{R} h_c(t) \, dt = \int_{-c}^{c} h_c(t) \, dt = 1.
\]

Thus, if \( 0 < c < R \), then every weight function \( h_c \) assigns the entire weight to the interval \( [-R,R] \), in the sense that \( \int_{a}^{b} h_c(t) \, dt = 0 \) for every interval \( [a,b] \subset ]-\infty,-R[ \cup ]R,\infty[ \).

The name “approximate identity” arises from the fact that averaging a function \( f \) with weight functions \( t \mapsto w_c(x - t) \) from an approximate identity approximates the value \( f(x) \) with increasing accuracy as \( c \) tends to zero, as shown in Figure 12.

![Figure 12](image-url)

Figure 12. The integral of the product of \( f \) and the weight function \( w \) gives a weighted average of the values \( f \) near the maximum of \( w \).
Proposition 1. For each function \( f : \mathbb{R} \rightarrow \mathbb{R} \) continuous at \( t = x \), and for each approximate identity \( \{ w_c : c > 0 \} \), we have

\[
f(x) = \lim_{c \to 0} \int_{\mathbb{R}} f(t) \cdot w_c(x - t) \, dt.
\]

Proof: The definition of continuity of \( f \) at \( x \) means that for each \( \varepsilon > 0 \) some \( \delta > 0 \) exists for which

\[
|f(t) - f(x)| < \varepsilon \quad \text{if} \quad |t - x| < \delta.
\]

Consequently, if \( 0 < c < \delta \) and if \( x - c < t < x + c \), then \( |t - x| < c < \delta \), whence

\[
\left| 1 \cdot f(x) - \int_{\mathbb{R}} f(t)w_c(x - t) \, dt \right| = \left| \int_{\mathbb{R}} f(x)w_c(t) \, dt - \int_{\mathbb{R}} f(t)w_c(x - t) \, dt \right|
\]

\[
= \left| \int_{\mathbb{R}} [f(x) - f(t)] w_c(x - t) \, dt \right|
\]

\[
\leq \int_{\mathbb{R}} |[f(x) - f(t)] w_c(x - t)| \, dt
\]

\[
< \int_{\mathbb{R}} \varepsilon |w_c(x - t)| \, dt
\]

\[
= \varepsilon \int_{\mathbb{R}} |w_c(x - t)| \, dt \leq \varepsilon M,
\]

which tends to zero as \( \varepsilon \) tends to zero. Thus,

\[
\lim_{c \to 0} \left| f(x) - \int_{\mathbb{R}} f(t)w_c(x - t) \, dt \right| = 0,
\]

\[
f(x) = \lim_{c \to 0} \int_{\mathbb{R}} f(t)w_c(x - t) \, dt.
\]

The integral of a product of shifted functions has a special name.

Definition 4. For each pair of functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( h : \mathbb{R} \rightarrow \mathbb{R} \) for which the following integral exists, define the convolution of \( f \) and \( h \) as the function denoted by \( f * h \) with

\[
(f * h)(r) := \int_{\mathbb{R}} f(s)h(r - s) \, ds.
\]

The following results show that any weight function may serve as the building block for an approximate identity.


**Proposition 2.** For each weight function $w : \mathbb{R} \to \mathbb{R}$ such that $w(t) = 0$ for every $t$ outside of $[-1, 1]$, the set $\{ w_c : c > 0 \}$ defined by

$$w_c(t) := \frac{1}{c} \cdot w \left( \frac{t}{c} \right)$$

constitutes an approximate identity.

**Proof:** Perform the change of variables $t := cs$ and $s := t/c$. For each $R > 0$, if $0 < c < R$, then

$$\int_{-c}^{c} w_c(t) \, dt = \int_{-c}^{c} \frac{1}{c} \cdot w \left( \frac{t}{c} \right) \, dt = \int_{-1}^{1} \frac{1}{c} w(s) \, c \, ds = \int_{-1}^{1} w(s) \, ds = 1.$$

The same change of variable confirms that for each $c$ but for the same $M > 0$,

$$\int_{-c}^{c} |w_c(t)| \, dt = \int_{-c}^{c} \left| \frac{1}{c} \cdot w \left( \frac{t}{c} \right) \right| \, dt = \int_{-1}^{1} \left| \frac{1}{c} \cdot w(s) \right| \, ds = \int_{-1}^{1} |w(s)| \, ds = M.$$

A similar result also holds for weight functions that need not equal zero outside such an interval as $[-1, 1]$, but the following material will not require such a result.

**Exercises**

25. (The function introduced here, known as the Gaussian distribution, plays an important role in the theory of Fourier transforms [Folland 1984, 243] and in probability [Folland 1984, 299].) Verify that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

is a weight function on $\mathbb{R}$. (Hint: consider the square of the integral as an integral over the plane and then express it with polar coordinates; alternatively, another method consists in verifying that the indefinite integral satisfies an ordinary differential equation [Weinstock 1990].) Then verify that the functions defined by

$$g_c(t) := \frac{1}{c} g \left( \frac{t}{c} \right) = \frac{1}{\sqrt{2\pi c^2}} e^{-t^2/2c^2}$$

form an approximate identity.
26. (The exponential function introduced here plays an important role in the
time of Fourier transforms [Hewitt and Stromberg 1965, 407 and 409].) Verify that the function \( K : \mathbb{R} \to \mathbb{R} \) defined by
\[
K(x) := \frac{1}{2} e^{-|x|}
\]
is a weight function on \( \mathbb{R} \), and consequently, that the functions defined by
\[
K_c(x) := \frac{1}{c} K \left( \frac{x}{c} \right) = \frac{1}{2c} e^{-|x/c|}
\]
form an approximate identity.

5.3 Approximate Identities with Several Dimensions

Definition 5. An \( n \)-dimensional weight function is a function \( w : \mathbb{R}^n \to \mathbb{R} \) such that \( \int_{\mathbb{R}^n} |w(\vec{x})| \, dx_1 \cdots dx_n < \infty \) and such that
\[
\int_{\mathbb{R}^n} w(\vec{x}) \, dx_1 \cdots dx_n = 1.
\]

Example 13. Consider the function \( f_{R,c} \) shown in Figure 13, with value \( f_{R,c} = c \left( 1 - \sqrt{x^2 + y^2}/R \right) \) in the closed disc \( D \) where \( x^2 + y^2 \leq R^2 \), and constant value 0 everywhere else:
\[
f_{R,c}(x, y) = \begin{cases} 
    c \left( 1 - \sqrt{x^2 + y^2}/R \right), & \text{if } x^2 + y^2 \leq R^2; \\
    0, & \text{if } x^2 + y^2 > R^2.
\end{cases}
\]
With the particular value \( c := 3/\pi R^2 \), polar coordinates confirm that \( f_{R,3/\pi R^2} \) is a weight function:
\[
\int_{\mathbb{R}^2} |f_{R,c}(x, y)| \, dx \, dy = \int_{\mathbb{R}^2} f_{R,c}(x, y) \, dx \, dy
\]
\[
= \int_D c \left( 1 - \sqrt{x^2 + y^2}/R \right) \, dx \, dy
\]
\[
= \int_{r=0}^{r=R} \int_{a=-\pi}^{a=\pi} c \left( 1 - \frac{r}{R} \right) \, r \, dr \, da
\]
\[
= \int_{r=0}^{r=R} 2\pi c \left( 1 - \frac{r}{R} \right) \, r \, dr
\]
\[
= \int_{r=0}^{r=R} 2\pi c \left( r - \frac{r^2}{R} \right) \, dr
\]
\[
= 2\pi c \left( \frac{r^2}{2} - \frac{r^3}{3R} \right) \bigg|_0^R = 2\pi c \left( \frac{R^2}{2} - \frac{R^3}{3R} \right)
\]
\[
= 2\pi c \frac{R^2}{6} = \frac{cR^2\pi}{3} = 1.
\]
Figure 13. The weight function $w$ defined by $w(x, y) = c \cdot \left( 1 - \frac{\sqrt{x^2 + y^2}}{R} \right)$.

Considerations similar to those for one dimension show that with several dimensions, weight functions also provide averages that approximate the value of continuous functions, as illustrated in Figure 14.

Figure 14. The multidimensional counterpart to Figure 12. The integral of the product of $f$ and the weight function $w$ gives a weighted average of the values $f$ near the maximum of $w$, where $w$ has nonzero values.

**Theorem 1.** For each function $f : \mathbb{R}^n \to \mathbb{R}$ continuous at $\bar{z} \in \mathbb{R}^n$, and for each weight function $w : \mathbb{R}^n \to \mathbb{R}$ (with the notation $\bar{x} = (x_1, \ldots, x_n)$),

$$f(\bar{z}) = \lim_{c \to 0} \int_{\mathbb{R}^n} f(\bar{x}) \cdot \frac{1}{c^n} \cdot w \left( \frac{1}{c} \cdot |\bar{z} - \bar{x}| \right) \, dx_1 \cdots dx_n.$$

**Proof:** With multiple integrals, mimic the proofs for one dimension. □
Exercises

27. Verify that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{align*}
g(x, y) := \begin{cases} 
0, & \text{if } x^2 + y^2 > 1; \\
\frac{1}{\pi}, & \text{if } x^2 + y^2 \leq 1
\end{cases}
\end{align*}$$

is a weight function.

28. Verify that the set of functions $\{g_c : c > 0\}$ with $g_c : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{align*}
g_c(x, y) := \begin{cases} 
0, & \text{if } x^2 + y^2 > c^2; \\
\frac{1}{\pi c^2}, & \text{if } x^2 + y^2 \leq c^2
\end{cases}
\end{align*}$$

is an approximate identity.

For the following exercises, let $\| \|$ denote the Euclidean norm, so that $\| \vec{z} \|^2 = z_1^2 + \cdots + z_n^2$ for each $\vec{z} = (z_1, \ldots, z_n)$.

![Figure 15. The bivariate Gaussian distribution with $g(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$.](image1)

![Figure 16. Level curves of the bivariate Gaussian distribution.](image2)

29. Verify that the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(\vec{z}) := \frac{1}{\sqrt{(2\pi)^n}} e^{-(\|\vec{z}\|^2)/2}$$

is a weight function on $\mathbb{R}^n$. Figures 15 and 16 display $g$ for $n = 2$. 
30. Verify that the functions defined by

\[ g_c(\mathbf{z}) := \frac{1}{\sqrt{(2\pi c^2)^n}} e^{-\frac{||\mathbf{z}||^2}{2c^2}} \]

form an approximate identity.

6. Inversion of the Radon Transform

This section explains the filtered back-projection technique to compute the yet unknown density values \( f(x, y) \) from the measurements \( (\mathcal{R}f)(r, a) \).

6.1 The Adjoint Radon Transform

Some of the first but unsuccessful attempts at using computers to solve for the values \( f(x, y) \) from the measurements \( (\mathcal{R}f)(r, a) \) relied on little theory. According to Shepp [1983], they computed an average of \( (\mathcal{R}f)(r, a) \) over all lines \( L_{r,a} \) through the point \( (x, y) \). Because the point \( (x, y) \) lies on the line \( L_{r,a} \) if but only if

\[ x \cos a + y \sin a = r, \]

it follows that for each point \( (x, y) \in \mathbb{R}^2 \) and for each angle \( a \in [0, \pi[ \) the line \( L_{r,a} \) passes through the point \( (x, y) \) if but only if

\[ r = x \cos a + y \sin a. \]

This formula parametrizes the subset of all straight lines that pass through the point \( (x, y) \) by specifying \( r \) for each \( a \). Consequently, the average value \( [\mathcal{R}^*(\mathcal{R}f)](x, y) \) of all the measurements along all such lines through \( (x, y) \) takes the form

\[ [\mathcal{R}^*(\mathcal{R}f)](x, y) = \frac{1}{\pi} \int_0^\pi (\mathcal{R}f)(x \cos a + y \sin a, a) \, da. \]

This formula seems intuitive, since it gives the average over all straight lines through the point of interest. However, regardless of the computing power available, it is useless because it does not conform to the theory outlined here [Shepp 1983]. Though the average takes into account all the measurements along all the lines through the point \( (x, y) \), it does not equal \( f(x, y) \), because it does not adequately cancel the contributions of other points along those lines. An exact formula requires a weighted average, over all lines in the plane, of the type

\[ \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\infty} (\mathcal{R}f)(x \cos a + y \sin a - r, a) \cdot G_c(r, a) \, dr \, da \]

with a weight function \( G_c \). Nevertheless, the function \( \mathcal{R}^* \) provides a first step in finding such a weight function \( G_c \) and hence in developing an exact method to compute \( f(x, y) \).
Definition 6. The adjoint of Radon’s transform is the function $\mathcal{R}^*$ that maps each continuous function $F$ defined on $L(R^2)$ to a function $\mathcal{R}^*F$ defined on $R^2$ by

$$(\mathcal{R}^*F)(x,y) := \frac{1}{\pi} \int_0^\pi F(x \cos a + y \sin a, a) \, da.$$ 

Figure 17. For $(x,y)$ on the horizontal axis, $|\langle(x,y), (\cos a, \sin a)\rangle| > 1$ if but only if $-\arccos(1/u) < a < \arccos(1/u)$, where $u := \sqrt{x^2 + y^2}$.

Example 14. On the set $L(R^2)$ of all straight lines $L_{r,a}$ in the plane, identified by their polar coordinates $r \in R$ and $a \in [0, \pi]$, consider the function $F$ defined by

$$F(r,a) := \begin{cases} 1, & \text{if } -1 \leq r \leq 1; \\ 0, & \text{if } 1 < |r|. \end{cases}$$

The following calculation of $\mathcal{R}^*F$ corresponds to the two cases in the definition of $F$. In the first case, if $x^2 + y^2 \leq 1$, then the Cauchy-Schwarz inequality gives

$$|x \cos a + y \sin a| = |\langle \vec{x}, \vec{a} \rangle| = \|\vec{x}\| \cdot \|\vec{a}\| \cdot |\cos (\angle(\vec{x}, \vec{a}))| \leq \|\vec{x}\| \cdot \|\vec{a}\| = \|\vec{x}\| \leq 1,$$

whence $-1 \leq x \cos a + y \sin a \leq 1$ and $F(x \cos a + y \sin a, a) = 1$, so that

$$(\mathcal{R}^*F)(x,y) = \frac{1}{\pi} \int_0^\pi F(x \cos a + y \sin a, a) \, da = \frac{1}{\pi} \int_0^\pi 1 \, da = 1.$$ 

In the second case, if $u^2 := x^2 + y^2 > 1$, then again two alternatives arise. If but only if the normal unit vector $\vec{a}$ and the vector $\vec{x} = (x,y)$ make an angle $\angle(\vec{x}, \vec{a})$ of magnitude less than $\arccos(1/u)$, so that $|\cos (\angle(\vec{x}, \vec{a}))| > 1/u = 1/\|\vec{x}\|$, as shown in Figure 17, then

$$|x \cos a + y \sin a| = |\langle \vec{x}, \vec{a} \rangle| = \|\vec{x}\| \cdot \|\vec{a}\| \cdot |\cos (\angle(\vec{x}, \vec{a}))| > 1,$$
whence \( F(x \cos a + y \sin a, \ a) = 0 \) by the definition of \( F \). In contrast, 
\( F(x \cos a + y \sin a, \ a) = 1 \) in the complementary arcs, where \( \text{Arccos} \left( \frac{1}{u} \right) \leq a \leq \pi - \text{Arccos} \left( \frac{1}{u} \right) \), whence

\[
(R^* F)(x, y) = \frac{1}{\pi} \int_0^\pi F(x \cos a + y \sin a, \ a) \, da \\
= \frac{1}{\pi} \int_{\text{Arccos} \left( \frac{1}{u} \right)}^{\pi - \text{Arccos} \left( \frac{1}{u} \right)} 1 \, da \\
= \frac{1}{\pi} (\pi - 2 \text{Arccos} \left( \frac{1}{u} \right)). \\
= \frac{2}{\pi} \text{Arccos} \left( \frac{1}{\sqrt{x^2 + y^2}} \right).
\]

Thus,

\[
(R^* F)(x, y) = \begin{cases} 
1, & \text{if } x^2 + y^2 \leq 1; \\
\frac{2}{\pi} \text{Arccos} \left( \frac{1}{\sqrt{x^2 + y^2}} \right), & \text{if } x^2 + y^2 > 1.
\end{cases}
\]

**Remark 1.** The function \( R^* \) maps a domain of functions to a set of functions, as do the ordinary and partial derivatives. For instance, let \( C^\infty(\mathbb{R}, \mathbb{R}) \) denote the set of all functions \( f : \mathbb{R} \to \mathbb{R} \) for which all derivatives \( f', f'', \ldots, f^{(k)}, \ldots \) exist; then the derivative \( D \) defined by \( Df = f' \) maps \( C^\infty(\mathbb{R}, \mathbb{R}) \) to \( C^\infty(\mathbb{R}, \mathbb{R}) \):

\[
D : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R}), \\
f \mapsto Df := f'.
\]

Similarly, let \( C^0(\mathcal{L}(\mathbb{R}^2), \mathbb{R}) \) denote the set of all continuous functions \( F \) defined on \( \mathcal{L}(\mathbb{R}^2) \), and let \( C^0(\mathbb{R}^2, \mathbb{R}) \) denote the set of all continuous functions defined on the plane \( \mathbb{R}^2 \); then \( R^* \) maps \( C^0(\mathcal{L}(\mathbb{R}^2), \mathbb{R}) \) to \( C^0(\mathbb{R}^2, \mathbb{R}) \):

\[
R^* : C^0(\mathcal{L}(\mathbb{R}^2), \mathbb{R}) \to C^0(\mathbb{R}^2, \mathbb{R}) \\
F \mapsto R^* F, \\
(R^* F)(x, y) = \frac{1}{\pi} \int_0^\pi F(x \cos a + y \sin a, \ a) \, da.
\]

The usefulness of the adjoint \( R^* \) lies in its ability to transform integrals on the abstract set \( \mathcal{L}(\mathbb{R}^2) \) to integrals on the plane \( \mathbb{R}^2 \), for integrals of the following type.

**Definition 7.** For all functions \( f, g : \mathbb{R}^2 \to \mathbb{R} \), define

\[
\langle f, g \rangle := \int_{\mathbb{R}^2} f(x, y) \cdot g(x, y) \, dx \, dy,
\]
provided that the integral exists.

Similarly, for all continuous functions $F, G : L(\mathbb{R}^2) \to \mathbb{R}$ define

$$\langle F, G \rangle := \frac{1}{\pi} \int_{a=0}^{a=\pi} \int_{-\infty}^{\infty} F(r, a) \cdot G(r, a) \, dr \, da.$$  

**Remark 2.** The operations $\langle , \rangle$ are inner products on vector spaces of functions. Their applications in computed tomography and elsewhere demonstrate the usefulness of abstract linear algebra.

The following proposition shows that the adjoint $R^*$ of $R$ has features similar to those of the transpose $A^T$ of a real matrix $A$, for which

$$\langle A\vec{x}, \vec{z} \rangle = \langle \vec{x}, A^T \vec{z} \rangle.$$  

**Proposition 3.** For each function $f : \mathbb{R}^2 \to \mathbb{R}$ continuous in some closed and bounded disc $D$ and equal to zero outside of $D$, and for each continuous function $F : L(\mathbb{R}^2) \to \mathbb{R}$,

$$\langle Rf, F \rangle = \langle f, R^*F \rangle.$$  

**Proof:** For each point $(x, y)$ in the plane $\mathbb{R}^2$, and for each straight line $L_{r,a}$ passing through $(x, y)$, perform a change to new coordinates $(r, s)$ with two new orthogonal axes through $(x, y)$: The first axis coincides with $L_{r,a}$, whereas the second axis lies perpendicularly to $L_{r,a}$ through $(x, y)$:

$$r = x \cos a + y \sin a,$$
$$s = -x \sin a + y \cos a.$$  

For use in a subsequent substitution, the inverse change of coordinates coincides with the parametrization of $L_{r,a}$:

$$x = r \cos a - s \sin a,$$
$$y = r \sin a + s \cos a.$$  

Because the Jacobian matrix has determinant equal to one, that is,

$$\det \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{pmatrix} = \det \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} = [\cos a]^2 + [\sin a]^2 = 1,$$

swapping the order of integration and then performing such a change of coordinates yields

$$\langle Rf, F \rangle$$
$$= \frac{1}{\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} (Rf)(r, a) \cdot F(r, a) \, dr \, da$$
$$= \frac{1}{\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(r \cos a - s \sin a, r \sin a + s \cos a) \, ds \right] \cdot F(r, a) \, dr \, da.$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{1}{\pi} \int_{0}^{\pi} f(r \cos a - s \sin a, r \sin a + s \cos a) \cdot F(r, a) \, da \right\} \, dr \, ds
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{1}{\pi} \int_{0}^{\pi} f(x, y) \cdot F(x \cos a + y \sin a, a) \, da \right\} \, dx \, dy
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ f(x, y) (\mathcal{R}^* F)(x, y) \} \, dx \, dy
= \langle f, \mathcal{R}^* F \rangle.

(Within multiple integrals, changes of coordinates and permutations of the order of integration require a justification. Such a justification holds for the continuous functions considered here but still involve a sophistication at the level of senior mathematical analysis [Fleming 1982; Folland 1984; Hewitt and Stromberg 1965].)

6.2 Filtered Back-Projection

The formula proposed here to solve for \( f(x, y) \) in terms of \( (\mathcal{R} f)(r, a) \) will involve translations (shifts) of a basic formula for the origin. Consequently, the following notation will prove useful.

**Definition 8.** For each function \( f : \mathbb{R}^2 \to \mathbb{R} \) and for each point \( \vec{u} = (u, v) \), define \( \vec{u} f \) through either of the formulas

\[
\vec{u} f(\vec{x}) := f(\vec{u} - \vec{x}),
\]

\[
(u,v) f(x, y) := f(u - x, v - y).
\]

Similarly, for each straight line \( L \subset \mathbb{R}^2 \), let \( \vec{u} - L \) denote the central symmetry of \( L - \vec{u} \) with respect to the origin:

\[
\vec{u} - L := \{ \vec{u} - \vec{x} : \vec{x} \in L \}.
\]

This notation provides an alternative formula for the weighted average of a continuous function \( f \) with a weight function \( g \):

\[
f(u, v) = \lim_{c \to 0} \int_{\mathbb{R}^2} f(x, y) \cdot \frac{1}{c} \cdot g \left( \frac{1}{c} [(u, v) - (x, y)] \right) \, dx \, dy
= \lim_{c \to 0} \int_{\mathbb{R}^2} f((u, v) - (x, y)) \cdot \frac{1}{c} \cdot g \left( \frac{1}{c} (x, y) \right) \, dx \, dy
= \lim_{c \to 0} (f * g_c)(u, v)
= \lim_{c \to 0} \langle [\vec{u} f], g_c \rangle
\]

The essential idea of filtered back-projection consists in finding a weight function \( G_c \) on the set of all lines \( \mathcal{L}(\mathbb{R}^2) \) such that \( g_c = \mathcal{R}^* G_c \), because then

\[
\langle [\vec{u} f], g_c \rangle = \langle [\vec{u} f], \mathcal{R}^* G_c \rangle = \langle \mathcal{R}(\vec{u} f), G_c \rangle,
\]
which then yields \( f(u, v) \) in terms of \( \mathcal{R}(\tilde{u}, f) \). The following result expresses \( \mathcal{R}(\tilde{u}, f) \) in terms of \( \mathcal{R} f \).

**Proposition 4.** \( [\mathcal{R}(\tilde{u}, f)](L) = (\mathcal{R} f)(\tilde{u} - L) \).

**Proof:** If \((X(s), Y(s))\) parametrizes \(L\), then \((u - X(s), v - Y(s))\) parametrizes \(\tilde{u} - L\). Hence,

\[
[\mathcal{R}(\tilde{u}, f)](L_{r,a}) = \int_{-\infty}^{\infty} [\tilde{u} f](r \cos a - s \sin a, r \sin a + s \cos a) \, ds
\]

\[
= \int_{-\infty}^{\infty} f(u - [r \cos a - s \sin a], v - [r \sin a + s \cos a]) \, ds
\]

\[
= (\mathcal{R} f)(\tilde{u} - L_{r,a}).
\]

These results yield the following formula for \( f(u, v) \) in terms of the measurements \((\mathcal{R} f)(L_{r,a})\) along all lines \(L_{r,a}\) in the plane:

\[
f(u, v) = \lim_{c \to 0} \langle [\tilde{u} f], g_c \rangle = \lim_{c \to 0} \langle [\tilde{u} f], \mathcal{R}^* G_c \rangle = \lim_{c \to 0} (\mathcal{R}[\tilde{u} f], G_c).
\]

Writing the term the farthest to the right explicitly gives

\[
f(u, v) = \lim_{c \to 0} \frac{1}{\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} \left((\mathcal{R} f)(u \cos a + v \sin a - r, a)\right) G_c(r) \, dr \, da. \tag{6}
\]

**Remark 3.** Some assumption on the nature of the function \( f \) is necessary, for instance, the hypotheses made earlier that \( f \) equals zero outside a bounded domain \( D \subset \mathbb{R}^2 \) [Armitage 1994].

**Remark 4.** Hardly anyone ever calculates by hand or symbolically with (6). Instead, the significance of (6) is that it provides a theory useful for the design of computer programs for computed tomography. In practice, computers associated with CT scanners receive measured values of \((\mathcal{R} f)(r_k, a_\ell)\) from the scanner and then approximate the tissue density at selected points \( f(u_m, v_n) \) with a numerical integration of the right-hand side of (6).

The selection of optimal sample values of \( r_k \) and \( a_\ell \) to produce the highest accuracy forms a subject of study in itself. Typical algorithms use evenly spaced directions \( a_\ell \) for \( \ell \in \{1, \ldots, p\} \) and a subset from evenly spaced distances \( r_k \) for \( k \in \{-q, \ldots, q\} \) in an optimal ratio \( p/q \approx \pi \), for example, \( p = 120 \) and \( q = 38 \), depending on the desired resolution of the final picture. Because such a Cartesian choice of polar coordinates does not correspond to a Cartesian grid of points \((u_m, v_n)\), however, additional intermediate interpolations and approximations become necessary. Some scanners use other indexing schemes, for instance, arranging x-rays not in bundles of parallel lines, but in fans emanating from a common source.

For a complete derivation of the theory, consult Natterer [1986a]; for an industrial-grade interactive demonstration, see Bossi and Kruse [1994]; and for a computer program with various algorithms, see Huesman et al. [1977].
Of course, the weight function $G_c$ depends on the weight function $g_c$. Users of computer-assisted symbolic systems may test their systems on the following two examples.

**Example 15.** A straightforward but lengthy calculation of $\mathcal{R}^*G_c$ [Niev-ergelt 1986], otherwise similar to that of **Example 14**, shows that if

$$G_c(r, a) = \begin{cases} \frac{1}{\pi c^2} \cdot 1, & \text{if } -c \leq r \leq c; \\ \frac{1}{\pi c^2} \left(1 - \frac{1}{\sqrt{1 - c^2/r^2}}\right), & \text{if } |r| > c, \end{cases}$$

then

$$(\mathcal{R}^*G_c)(x, y) = g_c(x, y) = \begin{cases} \frac{1}{\pi c^2}, & \text{if } x^2 + y^2 \leq c^2; \\ 0, & \text{if } x^2 + y^2 > c^2. \end{cases}$$

**Example 16.** A straightforward but lengthy calculation of $\mathcal{R}^*H_c$ [Schuster 1989], otherwise similar to that of **Example 14**, shows that if

$$H_c(r, a) = \begin{cases} \frac{3}{\pi c^2} \left(1 - \frac{\pi r}{2c}\right), & \text{if } -c \leq r \leq c; \\ \frac{3}{\pi c^2} \left(1 - \frac{r \text{Arcsin}(c/r)}{c}\right), & \text{if } |r| > c, \end{cases}$$

then

$$(\mathcal{R}^*H_c)(x, y) = h_c(x, y) = \begin{cases} \frac{3}{\pi c^2} \left(1 - \frac{\sqrt{x^2 + y^2}}{c}\right), & \text{if } x^2 + y^2 \leq c^2; \\ 0, & \text{if } x^2 + y^2 > c^2. \end{cases}$$

The weight function $G_c$ in **Example 15** involves only arithmetic operations, but it has vertical asymptotes at $r = \pm c$, which may cause difficulties in the numerical evaluation of the integrals. In contrast, the weight function $H_c$ in **Example 16** is continuous everywhere, without vertical asymptotes, but it requires the inverse trigonometric function $\text{Arcsin}$ in the computations of the integrals.

The following exercises demonstrate a method to design weight functions for filtered back-projection, by means of a concept introduced in the following definition.

**Definition 9.** For each continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ on $\mathbb{R}_+ = [0, \infty[$, define *Abel’s transform* of $f$ as the function $\mathcal{A}f : \mathbb{R}_+ \to \mathbb{R}$ expressed by the formula

$$(\mathcal{A}f)(v) := \int_0^v \frac{f(u) \, du}{\sqrt{v^2 - u^2}}.$$
(This formula conforms to Gardner [1995, 385–386], but variants exist [Gelfand and Gindikin 1990, 146–147].) Similarly, for each continuous function $F: \mathbb{R}_+ \to \mathbb{R}$ on $\mathbb{R}_+ = [0, \infty[$, define a function $BF: \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$(BF)(w) := \int_0^w \frac{F(v) \cdot v \, dv}{\sqrt{w^2 - v^2}}.$$ 

The exercises will show how Abel’s transform $A$ relates to Radon’s transform $R$, and how $B$ arises in the inversion of $A$.

**Exercises**

The use of an iterated integral and changes of variables to invert Abel’s transform resembles the same operations in the evaluation of the integral of the Gaussian distribution.

31. (Inversion of Abel’s transform)

a) Through a permutation of the order of integration with careful changes of limits of integration, supply the missing steps $[\cdots]$ in the proof that

$$\{B[Af]\}(w) = \int_{v=0}^{v=w} \frac{[Af](v) \cdot v \, dv}{\sqrt{w^2 - v^2}} = \cdots = \int_{u=0}^{u=w} f(u) \int_{v=u}^{v=w} \frac{v \, dv}{\sqrt{(w^2 - v^2)(v^2 - u^2)}} \, du.$$ 

b) By completing the square and through the change of variable

$$t := \frac{(2v^2 - [u^2 + v^2])}{(w^2 - u^2)},$$

prove that

$$\int_{v=0}^{v=w} \frac{[Af](v) \cdot v \, dv}{\sqrt{w^2 - v^2}} = \frac{\pi}{2} \int_{u=0}^{u=w} f(u) \, du.$$ 

c) Conclude by citing the logical step that guarantees that the following formula expresses $f$ in terms of $Af$:

$$\frac{2}{\pi} \{B[Af]\}'(w) = f(w).$$

32. (Relation between Abel’s and Radon’s transforms) Assume that $g: \mathbb{R}^2 \to \mathbb{R}$ and $G: \mathcal{L}(\mathbb{R}^2) \to \mathbb{R}$ are continuous even radial functions. This means that $g(x, y)$ depends on only $|v| = \sqrt{x^2 + y^2}$ and $G(u, a)$ depends on only $|u|$, so that $g(x, y) = q(v)$ and $G(u, a) = Q(u)$.

a) Prove that if $g = R^*G$, then

$$q(v) = \frac{2}{\pi} (AQ)(v).$$
b) Conclude that for each even radial weight function \( g \), the following formula gives the weight function \( G \) such that \( g = R^*G \):

\[
Q(w) = (Bq)'(w).
\]

33. With \( g_c \) as in Example 15, carry out the calculations in Exercise 32 to “discover” the required weight function \( G_c \).

34. With \( h_c \) as in Example 16, carry out the calculations in Exercise 32 to “discover” the required weight function \( H_c \).

7. Further Topics

This section briefly describes a few topics related to tomography.

7.1 Fan-Beam Tomography

After Cormack’s and Hounsfield’s first parallel-beam scanners, medical tomography switched to faster fan-beam scanners (Natterer [1986, 2]). In fan-beam scanners, the source \( S \) of x-rays moves along a fixed circle with radius \( R \) and center at the origin, and it emits x-rays in a fan pattern. Hence, if \( w \) denotes the polar angle of the source \( S \), which thus has polar coordinates \((w, R)\), and if \( c \) represents the angle of the x-rays, measured counterclockwise from the line through \( O \) and \( S \), then the two angles \( c \) and \( w \) may serve to identify the x-ray \( L \), instead of \( r \) and \( a \). To relate the two ways to identify x-rays, however, a transformation of coordinates becomes necessary.

35. Express \( r \) and \( a \) in terms of \( c \) and \( w \), and calculate the Jacobian determinant of the corresponding change of coordinates.

36. Insert the change of coordinates from the preceding exercise into the formula for filtered back-projection, to establish a corresponding numerical inversion of Radon’s transform for fan-beam tomography, in terms of \( c \) and \( w \) instead of \( r \) and \( a \).

7.2 Nuclear Magnetic Resonance (NMR)

NMR scanners emit not x-rays but electromagnetic pulses, and they measure the electromagnetic field caused by the nuclei of atoms along planes through tissue [Budinger and Gullberg 1974]. Thus, if \( f(x, y, z) \) denotes the density of the patient’s tissue at the location \((x, y, z)\), then for each plane \( P \) in space the NMR scanner measures the integral of \( f \) over \( P \),

\[
(\mathcal{R}f)(P) := \int_P f \, dA,
\]
Figure 18. On a fixed circle of radius $R$, the source $S$ has polar coordinates $(w, R)$ and emits an x-ray $L$ in a direction denoted by $c$. Thus $c$ and $w$ may serve to identity $L$ instead of $r$ and $a$. (All angles are measured counterclockwise.)

where $dA$ denotes the Euclidean measure of surface on the plane. The mathematical methods to solve for $f$ from the measurements $Rf$ include the use of approximate identities with three spatial variables and an adjoint transform [Natterer 1986, 8; Nievergelt 1991], but medical scanners also employ methods based on the Fourier transform [Natterer 1986, 8].

The following formulas from multivariable calculus lead to one of many methods to parametrize planes in space.

Solving for $\varphi$ and $\theta$ shows that for each vector $\vec{w} = (w_1, w_2, w_3)$ with unit length, there exist real numbers $\varphi$ and $\theta$ (the spherical coordinates of $\vec{w}$) for which

$$
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{pmatrix} =
\begin{pmatrix}
  \cos \varphi \sin \theta \\
  \sin \varphi \sin \theta \\
  \cos \theta
\end{pmatrix}.
$$

Moreover, for all real numbers $\varphi$ and $\theta$ with $\sin \theta \neq 0$, the following three vectors are orthonormal (mutually perpendicular with unit lengths):

$$
\vec{w} = \begin{pmatrix}
  \cos \varphi \sin \theta \\
  \sin \varphi \sin \theta \\
  \cos \theta
\end{pmatrix}, \quad \vec{u} = \begin{pmatrix}
  \cos \varphi \cos \theta \\
  \sin \varphi \cos \theta \\
  -\sin \theta
\end{pmatrix}, \quad \vec{v} = \begin{pmatrix}
  -\sin \varphi \\
  \cos \varphi \\
  0
\end{pmatrix}.
$$

Example 17. This example demonstrates how to parametrize a specific plane and how to set up the corresponding integral. Consider the plane $P_{r, \vec{w}}$ that lies at distance $r := \frac{3}{5}$ away from the origin in the direction of the unit vector $\vec{w} := \left( \frac{2}{11}, \frac{6}{11}, \frac{9}{11} \right)$. 
The first task consists in calculating two orthonormal vectors \( \vec{u} \) and \( \vec{v} \) parallel to the plane, or, equivalently, perpendicular to \( \vec{w} \). For instance, proceed with the spherical formulas above, or with cross products as in [Nievergelt 1992], or with the factorization \( \vec{w} = Q \cdot R \) [Kincaid and Cheney 1991]. All methods produce equivalent results, up to a rotation around the axis supporting \( \vec{w} \), for example, \( \vec{u} := \left( \frac{9}{11}, -\frac{6}{11}, \frac{2}{11} \right) \) and \( \vec{v} := \left( \frac{6}{11}, \frac{7}{11}, -\frac{6}{11} \right) \). Hence, for each point \( \vec{x} \in P_{r,\vec{w}} \), parameters \((p, q) \in \mathbb{R}^2\) exist such that

\[
\vec{x} = r \cdot \vec{w} + p \cdot \vec{u} + q \cdot \vec{v} = \left( \frac{3}{5} \right) \cdot \left( \begin{array}{c} \frac{2}{11} \\ \frac{6}{11} \\ \frac{9}{11} \end{array} \right) + p \cdot \left( \begin{array}{c} \frac{9}{11} \\ -\frac{6}{11} \\ \frac{2}{11} \end{array} \right) + q \cdot \left( \begin{array}{c} \frac{6}{11} \\ \frac{7}{11} \\ -\frac{6}{11} \end{array} \right).
\]

The second task involves calculating the Jacobian determinant of the parametrization, which, however, always equals 1 with orthonormal vectors:

\[
|\det(J)(p, q)| = \left\| \frac{\partial \vec{x}}{\partial p} \times \frac{\partial \vec{x}}{\partial q} \right\| = \|\vec{u} \times \vec{v}\| = \|\vec{w}\| = 1.
\]

Finally, the integral becomes

\[
(\mathcal{R}f)(P) = \int_P f \, dA = \int_{\mathbb{R}^2} f(r \cdot \vec{w} + p \cdot \vec{u} + q \cdot \vec{v}) \cdot |\det(J)(p, q)| \, dp \, dq \\
= \int_{\mathbb{R}^2} f(rw_1 + pu_1 + qv_1, rw_2 + pu_2 + qv_2, rw_3 + pu_3 + qv_3) \cdot 1 \, dp \, dq \\
= \int_{\mathbb{R}^2} f \left( r\left( \frac{2}{11} \right) + p\left( \frac{9}{11} \right) + q\left( \frac{6}{11} \right), r\left( \frac{6}{11} \right) + p\left( -\frac{6}{11} \right) + q\left( \frac{7}{11} \right), \right. \\
\left. \left( \frac{9}{11} \right) + p\left( \frac{2}{11} \right) + q\left( -\frac{6}{11} \right) \right) \cdot 1 \, dp \, dq.
\]

**Exercises**

37. For each real number \( r \) and each unit vector \( \vec{w} = (w_1, w_2, w_3) \), provide a method to parametrize the plane perpendicular to \( \vec{w} \) at distance \( r \) from the origin in the direction of \( \vec{w} \).

38. Parametrize the plane passing through \((2, 3, 6)\) and parallel to the vectors \((3, -6, 2)\) and \((6, 2, -3)\).

39. Design a method to specify the position of each plane in space \( \mathbb{R}^3 \). (Such a specification is necessary for NMR scanners, which measure electromagnetic radiations over planes through patients.)
40. Design a method to specify the position of each straight line in space \( \mathbb{R}^3 \). (Such a specification is necessary for CT scanners that use x-rays throughout a three-dimensional region.)

41. Consider the function \( f : \mathbb{R}^3 \to \mathbb{R} \) that assumes a constant value \( C \) in the ball \( B(\mathbf{0}, R) \) with radius \( R \) and center at the origin \( \mathbf{0} \), with the value 0 outside that ball. Also consider the plane \( P_{r, \hat{w}} \) that lies at distance \( r \) away from the origin in the direction \( \hat{w} \). Calculate the Radon transform of \( f \) for the plane \( P_{r, \hat{w}} \); in other words, derive a formula for \( (\mathcal{R}f)(P) = \int_P f \, dA \). 

42. Calculate the integral of the Gaussian distribution defined by

\[
g(x, y, z) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{(x^2+y^2+z^2)}{2}}
\]

over the plane \( P_{0, \hat{w}} \) that passes through the origin perpendicularly to the vector \( \hat{w} \).

7.3 Historical Notes on Computed Tomography

The mathematical literature on computed tomography—including the work of Nobel Prize laureate physicist Allan M. Cormack, whose work [Shepp 1983, 35] provides the basis of this account—credits the Dutch physicist Hendrik Antoon Lorentz (1853–1928) with the first theorem on computed tomography in a continuum (integrals, as opposed to linear systems). Though Lorentz does not seem to have published such a theorem, his credit comes from a publication by H.B.A. Bockwinkel in 1906, which attributes such a result to Lorentz and applies it to the propagation of light in biaxial crystals.

The literature also widely credits Johann Radon [1917] with first publication of a proof of the mathematical theorem that later became the crux of computed tomography under the name of “Radon’s Inversion Theorem.” Yet the application known as computed tomography did not exist in 1917. Thus, Radon proved his theory as a piece of abstract mathematics, without example, without application, and perhaps without foreseeing any application to tomography. Nevertheless, Radon describes relations between his theory, the Laplacian \( \Delta = D_1^2 + D_2^2 + D_3^2 \) of averages of functions along lines or planes, and the Newtonian potentials in electrostatics and gravity (in the plane, logarithmic; in space, according to the inverse-square law) [1917, 276–277]. Moreover, Radon also credits two physicists for antecedent work, specifically, Minkowski for reconstructing functions from their integrals along great circles on a sphere, and Funk for solving Minkowski’s problem with an alternate method that later provided the foundations for Radon’s own theory [Radon 1917, 275]. For a detailed account at the graduate level of the relations between integrals along lines, planes, circles, and spheres (generalized Radon transforms) and the partial differential equations of physics, consult John [1955].
The theory developed by Funk, Lorentz, Minkowski, and Radon yielded theoretical generalizations to higher dimensions by physicists Paul Ehrenfest and George Uhlenbeck in 1925, and to probability by Cramér and Wold in 1936 with the reconstruction of probability distributions from marginal distributions. Also in 1936, the Armenian astronomer V.A. Abartsumian independently rediscovered Radon’s theory and applied it to determine the distribution of velocities of stars from the measurements of their radial velocities. Abartsumian’s work apparently constitutes the first application of computed tomography, and, of course, in 1936, without computers.


In 1956, the radioastronomer R.N. Bracewell applied Radon’s theory to reconstruct the pattern of radio waves emanating from the sun from measurements of such radiations along thin strips.

Also in 1956, Cormack began his investigations of the application of Radon’s transform in medical radiology, which later earned him the joint Nobel Prize in medicine with G.N. Hounsfield, and led to CT scanners, NMR scanners, and thence much interplay between other applications and mathematics.

The history makes more precise the opening quotation of Lawrence A. Shepp: Much sophisticated mathematics has roots in applications, but from the applied problem much theoretical mathematics develops in abstracto before yielding a usable solution to the initial problem and to problems different from the initial problem.
8. **Sample Problems for Examinations**

The notation $L_{r,a}$ designates the straight line that lies at distance $r$ from the origin and that is perpendicular to the unit vector $\mathbf{a} = (\cos a, \sin a)$.

1. Calculate the Euclidean distance from the origin $\mathbf{0} = (0, 0)$ to a point

   $\mathbf{x} = r \cdot (\cos a, \sin a) + s \cdot (-\sin a, \cos a)$

   on the line $L_{r,a}$. Express your result as an algebraic formula in terms of $r$ and $s$.

2. Determine the Radon transform of the bivariate Gaussian distribution: calculate the line integral along $L_{r,a}$ of the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$g(x, y) = \frac{1}{2\pi} e^{-\left(x^2 + y^2\right)/2}.$$

   Express your result as a formula in terms of $r$ and elementary functions.

The notation $P_{r,\mathbf{w}}$ designates the plane that lies at distance $r$ from the origin $\mathbf{0}$ in the direction $\mathbf{w}$ and that is perpendicular to the unit vector $\mathbf{w}$.

3. Parametrize any plane at distance 9 from the origin and parallel to the two vectors $(4, 1, 8)$ and $(4, -8, -1)$.

4. Calculate the integral over the plane $P_{r,\mathbf{w}}$ of the function $G : \mathbb{R}^3 \to \mathbb{R}$ defined by

   $$G(x, y, z) = \frac{1}{(2\pi)^{3/2}} e^{-\left(x^2 + y^2 + z^2\right)/2}$$

   Express your result as a formula in terms of $r$ and elementary functions.

5. Design an algorithm to determine the intersection of the plane $P_{r,\mathbf{w}}$ and the straight line in space through a point $\mathbf{z}$ and parallel to a vector $\mathbf{c}$, with $\mathbf{c}$ not parallel to the plane. In other words, describe a finite number of operations with elementary functions that results in the three coordinates of the point of intersection.
References


Weinstock, Robert. 1990. Elementary evaluations of \( \int_{0}^{\infty} e^{-x^2} \, dx \), \( \int_{0}^{\infty} \cos(x^2) \, dx \), and \( \int_{0}^{\infty} \sin(x^2) \, dx \). *American Mathematical Monthly* 97 (1): 39–42.

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About the Author

Yves Nievergelt graduated in mathematics from the École Polytechnique Fédérale de Lausanne (Switzerland) in 1976, with concentrations in functional and numerical analysis of PDEs. He obtained a Ph.D. from the University of Washington in 1984, with a dissertation in several complex variables under the guidance of James R. King. He now teaches complex and numerical analysis at Eastern Washington University.

Prof. Nievergelt is an associate editor of The UMAP Journal. He is the author of several UMAP Modules, a bibliography of case studies of applications of lower-division mathematics (The UMAP Journal 6 (2) (1985): 37–56), and Mathematics in Business Administration (Irwin, 1989).