A Piecewise Linear Generalized Poisson Regression Approach to Modeling Longitudinal Frequency Data

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Outline

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Introduction

Problem: Inadequate modeling techniques for data from experiments that produce longitudinal frequency data.

Current Status: Different models have been fitted to such data (i.e. the Negative Binomial Distribution). These models are inadequate because they only allow for overdispersion.

Proposed Methodology: Use the Generalized Poisson Distribution to model data.
Generalized Poisson Distribution

The generalized Poisson distribution (see Consul and Jain 1973) is defined by the mass function

\[ p(x|\theta, \lambda) = \theta (\theta + x\lambda)^{x-1} e^{-(\theta + x\lambda)} \frac{1}{x!}, \quad x = 0, 1, 2, \ldots \]

for \( \theta > 0 \), and \( |\lambda| < 1 \), such that

\[ p(x|\theta, \lambda) = 0 \quad \text{for} \quad x \geq m \quad \text{when} \quad \lambda < 0; \]

\( m \) is the largest positive integer for which \( \theta + m\lambda \leq 0 \).
Generalized Poisson Distribution

It can be shown that if $X$ is a random variable with this generalized Poisson distribution then the expected value $\mu$ of $X$ is

$$\mu = \frac{\theta}{1 - \lambda}$$

with variance

$$\sigma^2 = \frac{\theta}{(1 - \lambda)^3}.$$ 

An alternative, more convenient parameterization of the generalized Poisson distribution is

$$X \sim GP(\mu, k),$$

where $\mu > 0$ is the expected value of the generalized Poisson random variable and $k$ is the dispersion parameter (Famoye and Wang 1997).
Generalized Poisson Distribution

The GP density $f(x|\mu, k)$ in terms of the parameters $\mu$ and $k$ is then

$$f(x|\mu, k) = \left( \frac{\mu}{1 + k\mu} \right)^x \frac{(1 + kx)^{x-1}}{x!} \exp \left( -\frac{\mu(1 + kx)}{1 + k\mu} \right); \quad x = 0, 1, 2, \ldots$$
Generalized Poisson Distribution

The variance of $X$, denoted by $VX$, is given by

$$VX = \mu(1 + k\mu)^2.$$  

- for $k > 0$ the variance of $X$ exceeds its expected value (overdispersion)
- for $\frac{-2}{\mu} < k < 0$ the expected value $\mu$ exceeds the variance of $X$ (underdispersion)
- for $k = 0$, $\mu = VX$, and the generalized Poisson distribution reduces to a standard Poisson distribution (equidispersion)
Univariate Parameter Estimation - begin by updating $\mu$ and $k$ as follows:

- Choose current (initial) values of the parameters, call these values $\mu^{(0)}$ and $k^{(0)}$.
- Propose a value for $\mu^*$ from a uniform distribution centered at $\mu^{(0)}$ the current value.

$$\mu^* \sim \text{Unif}(\mu^{(0)} - a, \mu^{(0)} + a),$$

where $a > 0$ is a tuning parameter in the proposal distribution for $\mu^*$. We have let the proposal distribution for $\mu^*$ be uniform which gives

$$q(\mu^* | \mu) = \frac{1}{\min(2a, \mu + a, \frac{1}{-2k})}.$$
Generalized Poisson Application

• For fixed $k$, the proposed value $\mu^*$ is then accepted with the following probability:

$$\alpha = \min \left(1, \frac{L(\mu^*, k)\pi(\mu^*, k)q(\mu^{(0)}|\mu^*)}{L(\mu^{(0)}, k)\pi(\mu^{(0)}, k)q(\mu^*|\mu^{(0)})}\right).$$

The value that is accepted for $\mu$— with probability $\alpha$— is then used to update $k$ in analogous fashion.

The values $\mu^{(1)}$ and $k^{(1)}$ represent the updated values of these parameters. These values are then treated as the new “current” parameter values, and the entire updating process is repeated.
The Equidispersed Data Set:

For the computer simulated equidispersed data set of $n = 200$ realizations:

- $\bar{x} = 7.03$
- $s^2 = 7.074472$
- $\hat{k} = 0.0004492$
Generalized Poisson Application

The Equidispersed Data Set:
The Underdispersed Data Set:

For the computer simulated underdispersed data set of \( n = 200 \) realizations:

- \( \bar{x} = 7.84 \)
- \( s^2 = 3.622513 \)
- \( \hat{k} = -0.0408486677 \)
Generalized Poisson Application

The Underdispersed Data Set:

![Graph showing iteration vs. values of μ and k](image-url)
Generalized Poisson Application

The Overdispersed Data Set:

For the computer simulated overdispersed data set of $n = 200$ realizations:

- $\bar{x} = 10.05$
- $s^2 = 17.50503$
- $\hat{k} = 0.03181794$
Generalized Poisson Application

The Overdispersed Data Set:
**Hot Flush Application**

**Data:** A clinical trial investigating acupuncture as an alternative treatment to alleviate symptoms of menopause for breast cancer survivors.

- Treatment group
- Placebo group
- Education group

![Graph](image)

Figure 1: The HFF profile of an individual from the treatment group.
Hot Flush Application

Data Model:

We model longitudinal frequency responses from multiple individuals using a GP distribution whose mean $\mu$ is a function of time.

The log-likelihood function $l_t(\mu_t, k)$ based on the $n_t$ observations collected at time $t$:

$$l_t(\mu_t, k) = \log \prod_{j=1}^{n_t} \frac{\mu_t^{y_{tj}} (1 + ky_{tj})^{y_{tj} - 1}}{1 + k\mu_t} \exp \left( -\mu_t \frac{(1 + ky_{tj})}{1 + k\mu_t} \right).$$
Flexibility in allowing $\mu_t$ to vary with time is obtained by modeling $\mu_t$ as a piecewise-linear function of time.

We define a vector of knot locations

$$K = \{K_1, K_2, \ldots, K_m\}$$

and a vector of corresponding heights

$$\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$$

$(K_i, \lambda_i)$ represents the location at which two line segments meet (referred to as a node).

$$\mu_t = f(\lambda_i, K_i, t) = \frac{\lambda_{i+1} - \lambda_i}{K_{i+1} - K_i} (t - K_i) + \lambda_i,$$

for $i = 1, \ldots, m$. 
Hot Flush Application

Five nodes were selected to formulate the piecewise linear function.

- three fixed knot locations \(\{0.5, 7.5, 91.5\}\)
- two additional knot locations randomly chosen in the time interval \((7.5, 91.5)\)

\[
K = \{K_1, K_2, \ldots, K_5\}
\]
\[
\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_5\}
\]

\(K_1 < K_2 < K_3 < K_4 < K_5\) and \(\lambda_i > 0\).

The complete list of parameters that defines this model is \(\{K, \lambda, k\}\), where \(k\) is the dispersion term.
Let 

\[ \{\lambda_1^{(i)}, \ldots, \lambda_5^{(i)}, K_3^{(i)}, K_4^{(i)}, k^{(i)} \} \]

represent the “current” value of the parameters.

- To update \( \lambda_1 \) let the proposal distribution be uniform over an interval centered around the current value \( \lambda_1^{(i)} \). Then

\[ \lambda_1^* \sim \text{Unif}(\lambda_1^{(i)} - a, \lambda_1^{(i)} + a) \]

where \( a \) is a tuning parameter. For fixed values of the other 7 parameters, accept \( \lambda_1^* \) as the next current value with probability \( \alpha \) given by

\[
\alpha = \min \left( 1, \frac{L(\lambda_1^*, \lambda_2^{(i)}, \ldots, \lambda_5^{(i)}, K^{(i)}, k^{(i)})\pi(\lambda_1^*)q(\lambda_1^{(i)}|\lambda_1^*)}{L(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_5^{(i)}, K^{(i)}, k^{(i)})\pi(\lambda_1^{(i)})q(\lambda_1^*|\lambda_1^{(i)})} \right).
\]

The other seven parameters are then updated using the “current” value of \( \lambda_1 \).
Hot Flush Application: Parameter Estimation

- The remaining node heights, $\lambda_2, \ldots, \lambda_5$, are updated in a manner analogous to updating $\lambda_1$.
- $K_3$ follows the same process as updating $\lambda_1$, but with the following exception: the proposed value of $K_3^*$ must be discrete uniform.
- $K_4$ is updated in a manner that is analogous to updating $K_3$.
- The dispersion parameter, $k$, is updated in the same way as $\lambda_1$.

After this cycle of updating the 8 parameters in turn is complete, it is repeated. This process is continued until the values stored for $\{\lambda^{(i+1)}, K^{(i+1)}, k^{(i+1)}\}$ have converged to the posterior distribution for $\{\lambda, K, k\}$. 
Trace plot of the marginal posterior draws for $k$ in the treatment group.

Trace plot of the marginal posterior draws for $k$ in the placebo group.

Trace plot of the marginal posterior draws for $k$ in the education group.
Hot Flush Application: Results

Treatment Group

Placebo Group

Education Group
**Discussion**

Usefulness of the **GP distribution:**

- Recognizes and treats the discrete nature of the data.
- Not only allows for, but detects equidispersed, underdispersed, and overdispersed data.

Usefulness of the **piecewise-linear model:**

- Ability to make comparisons across experimental groups.
- Allows for time correlation through the piecewise linear function for the mean.
- Useful in other applications involving longitudinal frequency data.
Limitations

The value chosen for $k$, determines the range of values that $\mu$ can take:

$$k = \frac{\lambda}{\theta}, \quad \mu = \frac{\theta}{1-\lambda}, \text{ and } -1 < \lambda < 0.$$  

By substitution,

$$-1 < k\theta < 0,$$
$$0 < \theta < -k.$$  

$\mu = \frac{\theta}{1-k\theta}$ over the interval for $\theta$ gives

$$0 < \mu < \frac{1}{-2k};$$

and

$$\frac{-1}{2\mu} < k < 0.$$
### Limitations

**Example:** Daily hot flush frequencies experienced by a subject in the placebo group.

- $\bar{x} = 13.73626$
- $s^2 = 1.640781$
- $\hat{k} = -0.0478$

Upper bound on $\mu$:

$$\frac{1}{-2\hat{k}} = \frac{1}{0.0956} = 10.46025.$$
**Future Work**

- Assign a mixture prior distribution for $k$ (i.e. allow $\pi(k = 0) \geq 0$).
- Compare with alternative longitudinal frequency models (i.e. a model that explicitly incorporates the dependence structure of the data).