REPRESENTATIONS BY QUADRATIC FORMS AND THE EICHLER COMMUTATION RELATION

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Let Q be a positive definite quadratic form on a lattice $L = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_m$, with associated symmetric bilinear form B so that $B(x,x) = Q(x)$. For convenience, we assume that L is even integral, meaning that $Q(L) \subseteq 2\mathbb{Z}$.

Question: Given an n-dimensional quadratic form T, on how many sublattices Λ of L does Q restrict to T?

To help us work toward an answer to this question, we define theta series, as follows.

Theta series:

For $n = 1$, and τ in the complex upper half-plane, we set

$$\theta(L;\tau) = \sum_{x \in L} \mathrm{e}\{Q(x)\tau\}$$

where $\mathrm{e}\{*\} = \exp(\pi i *)$. So

$$\theta(L;\tau) = \sum_{t \geq 0} r(L, 2t)\, \mathrm{e}\{2t\tau\}$$

where

$$r(L, 2t) = \#\{x \in L : \ Q(x) = 2t\ \}.$$

For $n > 1$, we want to make sense of

$$\sum_{\substack{\Lambda \subseteq L \\ \operatorname{rank}\Lambda \leq n}} \mathrm{e}\{\Lambda\tau\}.$$

Let $A = (B(x_i, x_j))$. With $C \in \mathbb{Z}^{m,n}$ and

$$(y_1 \cdots y_n) = (x_1 \cdots x_m)C,$$

let $\Lambda = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n$. So $\Lambda \subseteq L$ with $\operatorname{rank}\Lambda \leq n$ (notice that C need not have rank n), and ${}^tCAC = (B(y_i, y_j))$. So with $\mathrm{e}\{*\} = \exp(\pi i Tr(*))$, $\tau = X + iY$, $X, Y \in \mathbb{R}^{n,n}_{\mathrm{sym}}$ and $Y > 0$ (meaning that Y represents a positive definite quadratic form), we set

$$\theta^{(n)}(L;\tau) = \sum_{C \in \mathbb{Z}^{m,n}} \mathrm{e}\{\,{}^tCAC\tau\}.$$

To rewrite this more geometrically, take a lattice $\Lambda \subseteq L$ with $\operatorname{rank}\Lambda \leq n$, and take $y_1, \ldots, y_n$ to be vectors in L whose $\mathbb{Z}$-span is Λ. With $d = \operatorname{rank}\Lambda$, there is some $\mathbb{E} \in GL_n(\mathbb{Z})$ so that

$$(y_1 \cdots y_n)E = (v_1 \cdots v_d\, 0 \cdots 0).$$

Then

$${}^tE(B(y_i, y_j))E = T_\Lambda = \begin{pmatrix} T'_\Lambda & 0 \\ 0 & 0 \end{pmatrix}$$

where $T'_\Lambda = (B(v_i, v_j))$ (a $d \times d$ matrix that is invertible over $\mathbb{Q}$). Define

$$e\{\Lambda\tau\} = \sum_G e\{{}^tGT_\Lambda G\tau\}$$

where G varies over

$$\left\{\begin{pmatrix} I_d & 0 \\ * & * \end{pmatrix}\right\} \backslash GL_n(\mathbb{Z}).$$

(Note that $e\{\Lambda\tau\}$ is independent of the choice of basis $(v_1 \ldots v_d)$.) Then we have

$$\theta^{(n)}(L;\tau) = \sum_{\substack{\Lambda \in L \\ \operatorname{rank}\Lambda \le n}} e\{\Lambda\tau\},$$

and

$$\theta^{(n)}(L;\tau) = \sum_{\substack{\operatorname{cls}\Lambda \\ \operatorname{rank}\Lambda \le n}} r(L,\Lambda)\, e\{\Lambda\tau\}.$$

Modular forms and Hecke operators:

$\theta^{(n)}(L;\tau)$ is a Siegel modular form of degree n, weight $m/2$, some level N and quadratic character χ modulo N. This means that $\theta^{(n)}(L;\tau)$ is analytic (in all variables of τ), and

$$\theta^{(n)}(L;(A\tau+B)(C\tau+D)^{-1}) = \chi(\det D)\det(C\tau+D)^{m/2}\theta^{(n)}(L;\tau)$$

for all $(2n \times 2n)$ $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$ with $N|C$.

To avoid discussing the technical issues involved with taking square roots of complex numbers, from now on we assume that $m = 2k$ where k is an integer.

We know that for a prime p, $p \nmid N$ if and only if $\mathbb{Z}_pL$ is unimodular, and for $p \nmid N$, $\chi(p) = 1$ if and only if $\mathbb{Z}_pL$ is hyperbolic. (When $p \nmid N$, there are only 2 possible local structures for L, even when $p = 2$ as we have assumed that L is even integral.)

The Fourier coefficients of Hecke eigenforms satisfy arithmetic relations, so we want to explore how these operators act on our theta series. Ideally, we would like to describe the image of our theta series under a Hecke operator as a linear combination of theta series so that all these theta series lie in the same space of modular forms.

We first consider the case that $n = 1$. Take a prime p. Then with $T(p)$ the Hecke operator associated to p, we have

$$\theta(L;\tau)|T(p) = \sum_{\substack{x\in L \\ 2p|Q(x)}} e\{Q(x)\tau/p\} + \chi(p)p^{k-1}\sum_{x\in L} e\{Q(x)p\tau\}.$$

In the second sum, we replace $x \in L$ by $x \in pL$ to get

$$\theta(L;\tau)|T(p) = \sum_{\substack{x\in L \\ 2p|Q(x)}} e\{Q(x)\tau/p\} + \chi(p)p^{k-1}\sum_{x\in pL} e\{Q(x)\tau/p\}.$$

If $\chi(p) = 0$, we can't really hope to get a nice description of $\theta(L;\tau)|T(p)$ in terms of other theta series, so we assume $\chi(p) \neq 0$, and in fact we consider the case that $\chi(p) = 1$. (We later coment on the case $\chi(p) = -1$.) We proceed to sort the vectors $x \in L$ being counted in $\theta(L;\tau)|T(p)$ into sublattices of L corresponding to maximal totally isotropic subspaces of L/pL. To build these subspaces of L/pL, we begin by choosing an isotropic vector $v_1 \in L/pL$; we have $(p^k-1)(p^{k-1}+1)$ choices. Now we choose $u_1 \in L/pL$ so that $B(v_1,u_1) \neq 0$ (possible since L/pL is regular, in the sense that it has no radical). Thus $\langle v_1, u_1\rangle$ (the subspace spanned by v_1 and u_1) is a hyperbolic plane, and hence splits L/pL. Next, we choose an isotropic vector $v_2 \in \langle v_1\rangle^\perp \smallsetminus \langle v_1\rangle$; we have $p(p^{k-1}-1)(p^{k-2}+1)$ choices. Then we choose $u_2 \in \langle v_1,u_1\rangle^\perp$ so that $B(v_2,u_2)\neq 0$. So $\langle v_2,u_2\rangle$ is a hyperbolic plane. Continuing in this fashion, we build

$$[(p^k-1)(p^{k-1}+1)][p(p^{k-1}-1)(p^{k-2}+1)]\cdots[p^{k-1}(p^1-1)(p^0+1)]$$

bases for maximal totally isotropic subspaces of L/pL. As each dimension k space over $\mathbb{Z}/p\mathbb{Z}$ has

$$(p^k-1)(p^k-p)\cdots(p^k-p^{k-1})$$

bases, we build $(p^{k-1}+1)\beta$ maximal totally isotropic subspaces of L/pL, where

$$\beta = (p^{k-2}+1)\cdots(p^0+1).$$

(It is also easy to check that this gives us all the maximal totally isotropic subspaces of L/pL.) To count how many of these subspaces contain a given isotropic vector $v_1 \in L/pL$, we simply fix v_1 as the first basis vector in the above processes, showing that there are β maximal totally isotropic subspaces of L/pL that contain v_1.

(When $p=2$, this same procedure works, providing we replace Q on $L/2L$ by Q', being the quadratic form Q scaled by $1/2$, but we leave B as it was.)

Now, let K vary over the preimages in L of these maximal totally isotropic subspaces of L/pL. Thus

$$\begin{aligned}\sum_K \theta(K;\tau/p) &= \beta \sum_{\substack{x\in L\smallsetminus pL\\ 2p|Q(x)}} \mathrm{e}\{Q(x)\tau/p\} + (p^{k-1}+1)\beta\sum_{x\in pL}\mathrm{e}\{Q(x)\tau/p\}\\ &= \beta\sum_{\substack{x\in L\\ 2p|Q(x)}}\mathrm{e}\{Q(x)\tau/p\} + p^{k-1}\beta\sum_{x\in pL}\mathrm{e}\{Q(x)\tau/p\}\\ &= \beta\,\theta(L;\tau)|T(p).\end{aligned}$$

(This is often called the Eichler Commutation Relation.)

Now, what can we say about the local structures of one of these lattices K? Well, for any prime $q\neq p$, we have $\mathbb{Z}_qK = \mathbb{Z}_qL$. A basis $(v_1\ \cdots\ v_k\, u_1\ \cdots\ u_k)$ for L/pL that we constructed as above pulls back to a basis for L, and adjusting the u_i, we can ensure that

$$L \simeq \begin{pmatrix} 0 & I_k\\ I_k & *\end{pmatrix} \pmod{p}.$$

Then we get

$$K \simeq \begin{pmatrix} p* & pI_k\\ pI_k & 0\end{pmatrix} \pmod{p^2},$$

and consequently we have that $\mathbb{Z}_p K^{1/p}$ is hyperbolic (where $K^{1/p}$ means that we have scaled the quadratic form on K by $1/p$). Thus $\mathbb{Z}_p K^{1/p} \simeq \mathbb{Z}_p L$. However, we do not have $\mathbb{Z}_q K^{1/p} \simeq \mathbb{Z}_q L$ for $q \neq p$ if $\mathbb{Z}_q L$ has an odd rank Jordan component and p is not a square in $\mathbb{Z}_q$. So we may not have $K^{1/p} \in \operatorname{gen} L$. Still, these lattices K all lie in the same genus, which we call $\operatorname{gen} M$.

Now we sort these lattices K into isometry classes. We get

$$\begin{aligned}
&\beta\,\theta(L;\tau)|T(p)\\
&\quad= \sum_{\operatorname{cls} M' \in \operatorname{gen} M} \frac{\#\{\text{isometry } \sigma :\ pL \subseteq \sigma M' \subseteq L\ \}}{o(M')}\,\theta(M';\tau/p).
\end{aligned}$$

(Here $o(M')$ is the order of the orthogonal group of M'.) Next, we average over $\operatorname{gen} L$, to get

$$\begin{aligned}
&\beta\,\theta(\operatorname{gen} L;\tau)|T(p)\\
&= \beta \sum_{\substack{\operatorname{cls} L' \in \operatorname{gen} L\\ \operatorname{cls} M' \in \operatorname{gen} M}} \frac{\#\{\text{isometry } \sigma :\ pL' \subseteq \sigma M' \subseteq L'\ \}}{o(L')o(M')}\,\theta(M';\tau/p)\\
&= \beta \sum_{\operatorname{cls} M' \in \operatorname{gen} M} \left(\sum_{\operatorname{cls} L' \in \operatorname{gen} L} \frac{\#\{\sigma :\ pM' \subseteq \sigma pL' \subseteq M'\}}{o(L')} \right) \frac{\theta(M';\tau/p)}{o(M')}.
\end{aligned}$$

In this last expression, the inner sum over $\operatorname{cls} L' \in \operatorname{gen} L$ is counting the number of maximal totally isotropic subspaces of M'/pM' where the quadratic form on this space has been scaled by $1/p$. Thus we get

$$\theta(\operatorname{gen} L;\tau)|T(p) = (p^{k-1}+1)\theta(\operatorname{gen} M^{1/p};\tau).$$

So when $\operatorname{gen} L = \operatorname{gen} M^{1/p}$, $\theta(\operatorname{gen} L)$ is a $T(p)$-eigenform, and then we have the following relation on average representation numbers:

$$(p^{k-1}+1)\,r(\operatorname{gen} L, 2t) = r(\operatorname{gen} L, 2tp) + p^{k-1}\,r(\operatorname{gen} L, 2t/p).$$

Further, by the theory of modular forms, $\theta(\operatorname{gen} L)$ is known to lie in the space of Eisenstein series, and it is known that $\theta(\operatorname{gen} L) = \operatorname{mass} L \cdot \theta(L) + \text{cusp form}$. So

$$r(\operatorname{gen} L, 2t) \sim \operatorname{mass} L \cdot r(L, 2t) \text{ as } t \to \infty.$$

(Note: When $\operatorname{gen} L \neq \operatorname{gen} M^{1/p}$, it's not hard to see that $\theta(\operatorname{gen} L)|T(p)^2 = (p^{k-1}+1)^2\,\theta(\operatorname{gen} L)$. Also, when $\chi(p) = -1$, we can use a two-step construction of lattices sort of like K to get $\theta(\operatorname{gen} L)|T(p)^2 = (p^{k-1}-1)^2\,\theta(\operatorname{gen} L)$.)

We now consider the case $1 < n \le k$, and we still suppose that $\chi(p) = 1$. Then we have

$$\begin{aligned}\theta^{(n)}(L;\tau)|T(p) &= \sum_{\substack{\Lambda\subseteq L\\ \operatorname{rank}\Lambda\le n}} \left(\sum_{\substack{p\Lambda\subseteq\Omega\subseteq\Lambda\\ \Omega^{1/p}\text{ integral}}} p^{E(\Lambda,\Omega)}\, \mathrm{e}\{\Omega\tau/p\} \right)\\ &= \sum_{\substack{\Omega\subseteq L\\ \Omega^{1/p}\text{ integral}}} \left(\sum_{p\Omega\subseteq p\Lambda\subseteq\Omega} p^{E(\Lambda,\Omega)} \right) \mathrm{e}\{\Omega\tau/p\}\end{aligned}$$

where $\mathbb{E}(\Lambda,\Omega)$ is given by an explicit expression, depending on the index of Ω in Λ. To compute this inner sum, we note that those Λ with $[\Omega : p\Lambda] = p^d$ correspond to $n-d$-dimensional subspaces of $\Omega/p\Omega$. So we can easily count all these Λ. Contrastingly, we compare $\theta^{(n)}(L;\tau)|T(p)$ to $\theta^{(n)}(K;\tau/p)$, where K varies as in our discussion when n was 1. We count how often any given Ω lies in some K, and we find that

$$\theta^{(n)}(L;\tau)|T(p) = \gamma \sum_K \theta^{(n)}(K;\tau/p)$$

where $\gamma = (p^{k-n-1}+1)\cdots(p^0+1)$. Then averaging over the genus of L, the exact same argument as before gives us that

$$\theta^{(n)}(\operatorname{gen} L;\tau)|T(p) = (p^{k-1}+1)\cdots(p^{k-n}+1)\,\theta^{(n)}(\operatorname{gen} M^{1/p};\tau)$$

where $\operatorname{gen} M$ is as before (meaning that all these K lie in $\operatorname{gen} M$). So when $\operatorname{gen} L = \operatorname{gen} M^{1/p}$, we get

$$(p^{k-1}+1)\cdots(p^{k-n}+1)\, r(\operatorname{gen} L,\Lambda) = \sum_{p\Lambda\subseteq\Omega\subseteq\Lambda} p^{E(\Lambda,\Omega)}\, r(\operatorname{gen} L,\Omega^{1/p}).$$

But wait – there are more (algebraically independent) Hecke operators attached to p! For $1 \le j \le n$ with $j \le k$ if $\chi(p) = 1$, and $j < k$ if $\chi(p) = -1$, we compare $\theta(L;\tau)|T_j(p^2)$ to $\sum_{K_j} \theta(K_j;\tau)$ where $K_j \in \operatorname{gen} L$ where the invariant factors of K_j in L consist of j terms $1/p$, j terms p, and $2(k-j)$ terms 1.

We have

$$\begin{aligned}&\theta(L;\tau)|T_j(p^2)\\ &= \sum_{\Lambda\subseteq L} \left(\sum_{\substack{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda\\ \Omega\text{ integral}}} \chi(p^{e_j(\Lambda,\Omega)} p^{E_j(\Lambda,\Omega)} \alpha_j(\Lambda,\Omega)\, \mathrm{e}\{\Omega\tau\} \right).\end{aligned}$$

Here $e_j(\Lambda,\Omega), E_j(\Lambda,\Omega)$ are given in terms of the invariant factors of Ω in Λ, and $\alpha_j(\Lambda,\Omega)$ comes from an incomplete character sum. To complete the character sum, we replace $T_j(p^2)$ by $\widetilde{T}_j(p^2)$, which is a linear combination of $T_\ell(p^2)$ with $0 \le \ell \le j$ ($T_0(p^2)$ being the identity map). Then we get

$$\begin{aligned}
&\theta(L;\tau)|\widetilde{T}_j(p^2)\\
&= \sum_{\Lambda\subseteq L}\left(\sum_{\substack{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda\\ \Omega\text{ integral}}} \chi(p^{e_j(\Lambda,\Omega)}p^{E_j(\Lambda,\Omega)}\widetilde{\alpha}_j(\Lambda,\Omega)\,\mathrm{e}\{\Omega\tau\}\right)\\
&= \sum_{\substack{\Omega\subseteq\frac{1}{p}L\\ \Omega\text{ integral}}}\left(\sum_{p\Omega\subseteq\Lambda\subseteq\Delta} \chi(p^{e_j(\Lambda,\Omega)})p^{E_j(\Lambda,\Omega)}\widetilde{\alpha}_j(\Lambda,\Omega)\right)\mathrm{e}\{\Omega\tau\}
\end{aligned}$$

where $\Delta = \frac{1}{p}\Omega \cap L$, and with $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$, $\Omega = \frac{1}{p}\Lambda_0 \oplus \Lambda_1 \oplus p\Lambda_2$, $\widetilde{\alpha}_j(\Lambda,\Omega)$ is the number of codimension $n-j$ totally isotropic subspaces of $\Lambda_1/p\Lambda_1$.

To compare this to $\sum_{K_j}\theta(K_j;\tau)$, we count how often Ω is in some K_j. This will depend on the structure of Ω. We write $\Omega = \frac{1}{p}\Omega_0 \oplus \Omega'$ where Ω_0 is primitive in L modulo p (meaning that the rank of Ω_0 is the rank of the image of Ω_0 in L/pL). When $\operatorname{rank}\Omega_0 = i$, the coefficient of $\mathrm{e}\{\Omega\tau\}$ in $\theta(L;\tau)|\widetilde{T}_j(p^2)$ and in $\sum_{K_j}\theta(K_j;\tau)$ differ by a constant independent of the quadratic structure of Ω, but it's not the same constant for different i. So we adjust, finding constants so that

$$\theta(L;\tau)|T'_j(p^2) = \sum_{i=0}^{j} * \sum_{K_i}\theta(K_i;\tau)$$

where $T'_j(p^2)$ is a particular linear combination of $\widetilde{T}_i(p^2)$, $0 \le i \le j$. Then we average over the genus of L to get

$$\theta(\operatorname{gen} L)|T'_j(p^2) = p^*\beta(n,j)(p^{k-1}+\chi(p))\cdots(p^{k-j}+\chi(p))\theta(\operatorname{gen} L)$$

where $\beta(n,j)$ is the number of j-dimensional subspaces of an n-dimensional space over $\mathbb{Z}/p\mathbb{Z}$. This gives us relations on the average representation numbers, but they are complicated.

When $\chi(p) = 1$ and $j > k$, or when $\chi(p) = -1$ and $j \ge k$, we get

$$\theta(L;\tau)|T'_j(p^2) = 0.$$

We again have that $\theta(\operatorname{gen} L)$ is a linear combination of (Siegel) Eisenstein series, but using these to get formulas for the average representation numbers of L is currently out of reach, as we only have formulas for the Fourier coefficients of a basis for the space of Eisenstein series when the level is 1 and $k > n+1$, or when the degree is 2, $k > 3$, the level is square-free, and the character is trivial.

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