Generating weights for modules of vector-valued modular forms

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Wesleyan University, June 3rd, 2016

The metaplectic group

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- Generators: $T := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right), \quad S := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$

Vector valued modular forms

Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be complex, finite-dimensional representation.

Definition

A ρ-valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function $f : \mathfrak{h} \to V$ such that

$$f(\gamma\tau) = \phi^{2k}\rho(M)\, f(\tau)$$

for all $M = (\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

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- Growth conditions at ∞ are specified by a matrix L such that $\rho(T) = e^{2\pi i L}$
- Denote by $M_k(\rho)$ (resp. $S_k(\rho)$) the space of holomorphic modular forms (resp. cusp forms).

The free-module Theorem

Let

$$M(\rho) := \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} M_k(\rho),$$

viewed as a module over $M(1) = \mathbb{C}[E_4, E_6]$.

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Theorem (C., Franc, 2015)

(i) *$M(\rho)$ is a free module of rank $n = \dim \rho$ over $M(1)$.*

(ii) *If $k_1 \leq \ldots \leq k_n$, $k_j \in \frac{1}{2}\mathbb{Z}$, are the weights of the free generators, then*

$$\sum_{j=1}^{n} k_j = 12\,\mathrm{Tr}(L).$$

(iii) *If ρ is unitary, then $0 \leq k_j \leq 23/2$.*

Finding the generating weights

Main Question

Given ρ, find the weights $k_1, \ldots, k_n$ of the generators $M(\rho)$, the generating weights of $M(\rho)$.

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- We have

$$\sum_{k \in \frac{1}{2}\mathbb{Z}} \dim M_k(\rho) t^k = \frac{t^{k_1} + \ldots + t^{k_n}}{(1-t^4)(1-t^6)} \in \mathbb{Z}[\![t^{1/2}]\!]$$

so the question is equivalent to finding $\dim M_k(\rho)$ for all k.

Finite quadratic modules

Definition

A finite quadratic module is a pair (D, q) of a finite abelian group D together with a quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, whose associated bilinear form we denote by $b(x, y) := q(x + y) - q(x) - q(y)$.

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E.g.

For $m > 0$ even, let $\Lambda = (\mathbb{Z}, x \mapsto \frac{m}{2}x^2)$, a rank 1 lattice. The discriminant form of Λ is the finite quadratic module

$$A_m := (\mathbb{Z}/m\mathbb{Z}, x \mapsto \frac{x^2}{2m})$$

The Weil Representation

Let (D, q) be a finite quadratic module. Let $\mathbb{C}(D)$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x \in D}$ of delta functions, i.e. $\delta_x(y) = \delta_{x,y}$.

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Definition

The Weil representation $\rho_D : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}(D))$ is defined with respect to the basis $\{\delta_x\}_{x \in D}$ by

$$\rho_D(T)(\delta_x) = e^{-2\pi i\, q(x)} \delta_x$$

$$\rho_D(S)(\delta_x) = \frac{\sqrt{i}^{\mathrm{sig}(D)}}{\sqrt{|D|}} \sum_{y \in D} e^{2\pi i\, b(x,y)} \delta_y,$$

where $\frac{1}{\sqrt{|D|}} \sum_{x \in D} e^{2\pi i q(x)} = \sqrt{i}^{\mathrm{sig}(D)}$.

Generating weights of Weil representations

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For $D = A_m$, $k \in \frac{1}{2} + \mathbb{Z}$, we have

$$M_k(\rho_{A_m}) \simeq J_{k+1/2,m/2},$$

i.e. Jacobi forms of weight $k + 1/2$, index $m/2$ and

$$M(\rho_D) \simeq J_{m/2}$$

the (free) $\mathbb{C}[E_4, E_6]$-module of Jacobi forms of index $m/2$.

Attempt to compute dim $M_k(\rho_D)$ via Riemann-Roch

- Form the vector bundle $\mathcal{W}_k(\rho_D)$ over $\overline{\mathcal{M}_{1/2}} = \overline{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathfrak{h}}$.

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Theorem (Case A_{2p}, $p > 3$ prime, $k \in 1/2 + \mathbb{Z}$)

Let L_p such that $e^{2\pi i L_p} = \rho_{A_{2p}}(T)$, and with eigenvalues in $[0, 1)$.

$$\chi(\mathcal{W}_k(\rho_{A_{2p}})) = \frac{5+k}{12}p - \frac{1}{2}\operatorname{Tr}(L_p) + (-1)^{2k}(\delta + \frac{5+k}{12}) + \epsilon_\pm$$

Here

$$\delta := \frac{1}{8}\left(2 + \left(\frac{-1}{p}\right)\right), \quad \epsilon_\pm := \frac{1}{6}\left(1 \pm \left(\frac{p}{3}\right)\right)$$

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(iv) $M_{1/2}(\rho) = 0$ (Serre-Stark, Skoruppa)

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(v) $\dim M_{3/2}(\rho) = \chi(\mathcal{W}_{3/2}(\rho))$ (i.e. $H^1 = 0$, Skoruppa).

Computations

E.g.

For $p = 5$, the generating weights for $M(\rho_{A_{10}})$ are

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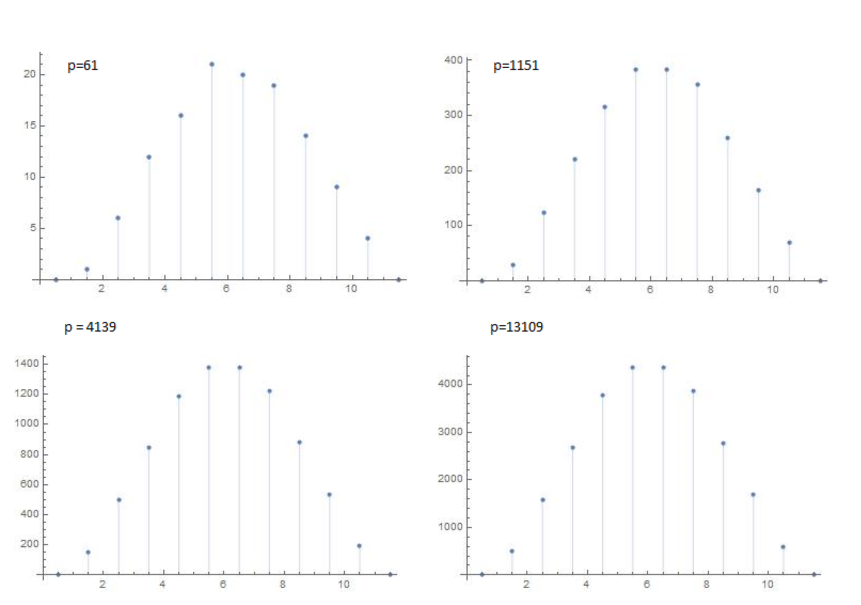
Try some larger primes $p = 61, 1151, 4139, 13109, \ldots$


p=61
p=1151
p = 4139
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Generating weights for $m = 2p$, $p \geq 5$

weight $= 1/2$	*multiplicity* $= 0$
$3/2$	$\frac{13}{24}(p+1) - \frac{1}{2}\operatorname{Tr}(L_p) - \delta - \epsilon_+$
$5/2$	$\frac{15}{24}(p-1) - \frac{1}{2}\operatorname{Tr}(L_p) + \delta$
$7/2$	$\frac{17}{24}(p+1) - \frac{1}{2}\operatorname{Tr}(L_p) - \delta + \epsilon_+$
$9/2$	$\frac{19}{24}(p-1) - \frac{1}{2}\operatorname{Tr}(L_p) + \delta + \epsilon_-$
$11/2$	$\frac{1}{3}(p+1) + \epsilon_+$
$13/2$	$\frac{1}{3}(p-1) - \epsilon_-$
$15/2$	$\frac{-5}{24}(p+1) + \frac{1}{2}\operatorname{Tr}(L_p) + \delta - \epsilon_+$
$17/2$	$\frac{-7}{24}(p-1) + \frac{1}{2}\operatorname{Tr}(L_p) - \delta - \epsilon_-$
$19/2$	$\frac{-9}{24}(p+1) + \frac{1}{2}\operatorname{Tr}(L_p) + \delta$
$21/2$	$\frac{-11}{24}(p-1) + \frac{1}{2}\operatorname{Tr}(L_p) - \delta + \epsilon_-$
$23/2$	0

Distribution as $p \to \infty$

Theorem (C., Franc, Kopp, 2016)

Let p be an odd prime and let $m = 2p$, $p > 3$. Then

$$\mathrm{Tr}(L_p) = \begin{cases} p + \frac{1}{2}h_p - \frac{1}{4} & p \equiv 1 \pmod 4, \\ p + 2h_p - \frac{1}{4} & p \equiv 3 \pmod 8, \\ p + h_p - \frac{1}{4} & p \equiv 7 \pmod 8. \end{cases}$$

where h_p is the class number of $\mathbb{Q}(\sqrt{-p})$.

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Corollary

If $\rho = \rho_{A_{2p}}$, then

$$\frac{\mathrm{Tr}(L_p)}{2p} \to 1/2, \quad p \to \infty.$$

Distribution of weights for $m = 2p$, as $p \to \infty$

$weight = 1/2$	$proportion = 0$
$3/2$	$1/48$
$5/2$	$3/48$
$7/2$	$5/48$
$9/2$	$7/48$
$11/2$	$8/48$
$13/2$	$8/48$
$15/2$	$7/48$
$17/2$	$5/48$
$19/2$	$3/48$
$21/2$	$1/48$
$23/2$	0

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Moral of the story

Generating weights might be easier to study than dimensions!