Generating weights for modules of vector-valued modular forms

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LSU

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The metaplectic group

- The **metaplectic group** $\text{Mp}_2(\mathbb{Z})$ is the unique nontrivial central extension

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- Multiplication:

\[ (A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1 A_2, \phi_1(A_2 \tau) \phi_2(\tau)). \]
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- Generators: \( T := ((1 1) , 1), S := ((0 -1) , \sqrt{\tau}) \)
Vector valued modular forms

Let $\rho : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(V)$ be complex, finite-dimensional representation.

**Definition**

A $\rho$-valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ is a holomorphic function $f : \mathfrak{h} \to V$ such that

$$f(\gamma \tau) = \phi^{2k} \rho(M) f(\tau)$$

for all $M = (\gamma, \phi) \in \text{Mp}_2(\mathbb{Z})$. 

- Growth conditions at $\infty$ are specified by a matrix $L$ such that $\rho(T) = e^{2\pi i L}$. 

Denote by $M_k(\rho)$ (resp. $S_k(\rho)$) the space of holomorphic modular forms (resp. cusp forms).
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The free-module Theorem

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\[ M(\rho) := \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} M_k(\rho), \]

viewed as a module over \( M(1) = \mathbb{C}[E_4, E_6] \).
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**Theorem (C., Franc, 2015)**

(i) $M(\rho)$ is a free module of rank $n = \dim \rho$ over $M(1)$.

(ii) If $k_1 \leq \ldots \leq k_n$, $k_j \in \frac{1}{2} \mathbb{Z}$, are the weights of the free generators, then

$$\sum_{j=1}^{n} k_j = 12 \text{ Tr}(L).$$

(iii) If $\rho$ is unitary, then $0 \leq k_j \leq 23/2$. 

Finding the generating weights

Main Question

Given $\rho$, find the weights $k_1, \ldots, k_n$ of the generators $M(\rho)$, the generating weights of $M(\rho)$.
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- We have

$$\sum_{k \in \frac{1}{2} \mathbb{Z}} \dim M_k(\rho) t^k = \frac{t^{k_1} + \ldots + t^{k_n}}{(1 - t^4)(1 - t^6)} \in \mathbb{Z}[t^{1/2}]$$

so the question is equivalent to finding $\dim M_k(\rho)$ for all $k$. 
Finite quadratic modules

**Definition**

A **finite quadratic module** is a pair \((D, q)\) of a finite abelian group \(D\) together with a quadratic form \(q : D \to \mathbb{Q}/\mathbb{Z}\), whose associated bilinear form we denote by \(b(x, y) := q(x + y) - q(x) - q(y)\).
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E.g.

For \(m > 0\) even, let \(\Lambda = (\mathbb{Z}, x \mapsto \frac{m}{2}x^2)\), a rank 1 lattice. The discriminant form of \(\Lambda\) is the finite quadratic module

\[
A_m := (\mathbb{Z}/m\mathbb{Z}, x \mapsto \frac{x^2}{2m})
\]
The Weil Representation

Let $(D, q)$ be a finite quadratic module. Let $\mathbb{C}(D)$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x \in D}$ of delta functions, i.e. $\delta_x(y) = \delta_{x,y}$. 
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**Definition**

The **Weil representation** \(\rho_D : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(\mathbb{C}(D))\) is defined with respect to the basis \(\{\delta_x\}_{x \in D}\) by

\[
\rho_D(T)(\delta_x) = e^{-2\pi i q(x)} \delta_x
\]

\[
\rho_D(S)(\delta_x) = \frac{\sqrt{-\text{sig}(D)}}{\sqrt{|D|}} \sum_{y \in D} e^{2\pi i b(x,y)} \delta_y,
\]

where \(\frac{1}{\sqrt{|D|}} \sum_{x \in D} e^{2\pi i q(x)} = \sqrt{i}^{\text{sig}(D)}\).
Generating weights of Weil representations

Main Question

Given $D$, find the generating weights of $M(\rho_D)$.

(Equivalent to finding $\text{dim } M_k(\rho_D)$ for all $k \in \frac{1}{2}\mathbb{Z}$).
Generating weights of Weil representations

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(Equivalent to finding $\dim M_k(\rho_D)$ for all $k \in \frac{1}{2} \mathbb{Z}$).

E.g.

For $D = A_m$, $k \in \frac{1}{2} + \mathbb{Z}$, we have

$$M_k(\rho_{A_m}) \simeq J_{k+1/2, m/2},$$

i.e. Jacobi forms of weight $k + 1/2$, index $m/2$ and

$$M(\rho_D) \simeq J_{m/2}$$

the (free) $\mathbb{C}[E_4, E_6]$-module of Jacobi forms of index $m/2$. 
Attempt to compute $\dim M_k(\rho_D)$ via Riemann-Roch

- Form the vector bundle $\mathcal{W}_k(\rho_D)$ over $\mathcal{M}_{1/2} = \overline{M_{p_2}(\mathbb{Z}) \setminus \mathfrak{h}}$. 

\[ \chi(\mathcal{W}_k(\rho_{A_2p})) = \dim M_k(\rho_{A_2p}) - \dim \mathcal{H}_1(\mathcal{W}_k(\rho_{A_2p})) \]

- When is the $\mathcal{H}_1$ term zero?

**Theorem (Case $A_{2p}$, $p > 3$ prime, $k \in 1/2 + \mathbb{Z}$)**

Let $L_p$ such that $e^{2\pi i L_p} = \rho_{A_2p}(T)$, and with eigenvalues in $[0, 1)$.

\[ \chi(\mathcal{W}_k(\rho_{A_2p})) = 5 + k - \frac{1}{2} \text{Tr}(L_p) + (-1)^{2k} \left( \delta + 5 + \frac{k}{12} \right) + \epsilon_{\pm} \]

Here $\delta = \frac{1}{8} (2 + (-1)^{p/3})$, $\epsilon_{\pm} = \frac{1}{6} (1 \pm (-1)^p)$.
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\chi(\mathcal{W}_k(\rho_{A_{2p}})) = \frac{5 + k}{12} p - \frac{1}{2} \text{Tr}(L_p) + (-1)^{2k}(\delta + \frac{5 + k}{12}) + \epsilon_{\pm}
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\[
\delta := \frac{1}{8} \left( 2 + \left( -\frac{1}{p} \right) \right), \quad \epsilon_{\pm} := \frac{1}{6} \left( 1 \pm \left( \frac{p}{3} \right) \right)
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For $\rho = \rho_{A_{2p}}$, $p$ prime, we have:

(i) $M_k(\rho) = 0$ if $k < 0$. 

(ii) $M_k(\rho) = 0$ if $k \in \mathbb{Z}$. 

(iii) $\dim M_k(\rho) = \chi(W_k(\rho))$, $k > 3/2$. 

(iv) $M_{1/2}(\rho) = 0$ (Serre-Stark, Skoruppa). 

(v) $\dim M_{3/2}(\rho) = \chi(W_{3/2}(\rho))$ (i.e. $H_1 = 0$, Skoruppa).
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E.g.

For $p = 5$, the generating weights for $M(\rho_{A_{10}})$ are

$$\frac{1}{2} (7, 9, 11, 11, 13, 15, 15, 17, 19, 21)$$
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Try some larger primes $p = 61, 1151, 4139, 13109, ...$
## Generating weights for $m = 2p$, $p \geq 5$

<table>
<thead>
<tr>
<th>weight</th>
<th>multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
<td>$0$</td>
</tr>
<tr>
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<td>$\frac{13}{24} (p + 1) - \frac{1}{2} \text{Tr}(L_p) - \delta - \epsilon_+$</td>
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<tr>
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</tr>
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<tr>
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**Theorem (C., Franc, Kopp, 2016)**

Let $p$ be an odd prime and let $m = 2p$, $p > 3$. Then

\[
\text{Tr}(L_p) = \begin{cases} 
  p + \frac{1}{2} h_p - \frac{1}{4} & \text{if } p \equiv 1 \pmod{4}, \\
  p + 2h_p - \frac{1}{4} & \text{if } p \equiv 3 \pmod{8}, \\
  p + h_p - \frac{1}{4} & \text{if } p \equiv 7 \pmod{8}.
\end{cases}
\]

where $h_p$ is the class number of $\mathbb{Q}(\sqrt{-p})$. 

**Corollary**

If $\rho = \rho_{A,2p}$, then

\[
\text{Tr}(L_p)^{2p} \to \frac{1}{2}, \quad p \to \infty.
\]

**Heuristic:** $\text{Tr}(L_p) \approx \frac{1}{2} \dim(\rho)$. 


Distribution as $p \to \infty$

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**Corollary**

If $\rho = \rho_{A_{2p}}$, then

$$\frac{\text{Tr}(L_p)}{2p} \to \frac{1}{2}, \quad p \to \infty.$$
Distribution of weights for $m = 2p$, as $p \to \infty$

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Moral of the story

Generating weights might be easier to study than dimensions!