Homework #4

1. Let $p$ be an odd prime. Prove that
   
   (a) $1^{p-1} + 2^{p-1} + \ldots + (p-1)^{p-1} \equiv -1 \mod p$

   (b) $1^p + 2^p + \ldots + (p-1)^p \equiv 0 \mod p$

2. If $m$ and $n$ are relatively prime positive integers, prove that

   $$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \mod mn.$$ 

3. Find all positive integers $n$ such that

   (a) $\phi(n) = 16$

   (b) $\frac{\phi(n)}{n} = \frac{2}{3}$

4. For natural numbers $a, b$ and $n$, suppose that $ord_n(a) = h$ and $ord_n(b) = k$. Show that the $ord_n(ab)$ divides $hk$. Use this to show that if $\gcd(h, k) = 1$ then $ord_n(ab) = hk$. 
