Homework #6

1. Find all positive integers n such that

$$\frac{\phi(n)}{n} = \frac{2}{3}.$$

2. Prove the following:

   (a) If n and $n+2$ are a pair of twin primes, then $\phi(n+2) = \phi(n) + 2$.

   (b) If p and $2p+1$ are both odd primes, then $n = 4p$ satisfies $\phi(n+2) = \phi(n) + 2$.

3. (a) If $\phi(n) \mid (n-1)$, prove that n is a square-free integer.

   (b) Show that if $n = 2^k$ or $2^k 3^j$ with k, j positive integers, then $\phi(n) \mid n$.

4. Find all positive integers n such that $\phi(n) = 16$.

5. Assume that the order of $a \mid n$ is h and the order of $b \mid n$ is k. Show that the order of $ab \mod n$ divides hk; in particular, if $\gcd(h, k) = 1$, then ab has order hk.

6. (a) Find a primitive root $\mod 23$

   (b) For each positive divisor d of 22, find the number of integers among 1,...,22 whose orders $\mod 23$ are d and then verify the equality

$$\sum_{d|22} \phi(d) = 22$$

   (c) Let g be the primitive root you found in (a). Use it to express every integer from 1 to 22 as a power of a.

   (d) Use the data you obtain form (c) to decide whether the congruence $x^5 \equiv 7 \mod 23$ has a solution. If it does, find all the solutions $\mod 23$.

7. Let p be an odd prime and g be a primitive root $\mod p$.

   (a) Show that $g^{(p-1)/2} \equiv -1 \mod p$

   (b) By expressing every integer from 1 to $p-1$ as a power of g, give another proof of Wilson's theorem.

8. Let p be an odd prime and g be a primitive root $\mod p$. Prove the following:

   (a) If $p \equiv 1 \mod 4$, then $-g$ is also a primitive root $\mod p$.

   (b) If $p \equiv 3 \mod 4$, then $-g$ has order $(p-1)/2 \mod p$.