Math 311 Fall 2015

Homework #5

1. Let p be an odd prime. Prove that
   (a) $1^{p-1} + 2^{p-1} + ... + (p-1)^{p-1} \equiv -1 \mod p$
   (b) $1^p + 2^p + ... + (p-1)^p \equiv 0 \mod p$

2. Prove that $1835^{1910} + 1986^{2061} \equiv 0 \mod 7$.

3. Let p be an odd prime and k an integer between a and $p-1$. Prove that

$$\binom{p-1}{k} \equiv (-1)^k \mod p.$$

4. Prove that for any odd prime p

$$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \mod p$$

5. (a) For a prime p of the form $4k+3$, prove that either

$$\left(\frac{p-1}{2}\right)! \equiv 1 \mod p \text{ or } \left(\frac{p-1}{2}\right)! \equiv -1 \mod p$$

   hence $[(p-1)/2]!$ is a solution to the congruence $x^2 \equiv 1 \mod p$.

   (b) Use part (a) to show that if $p = 4k+3$ is a prime, then the product of all the even positive integers less than p is congruent $\mod p$ to either -1 or 1.

   (c) Find the unit digit of 3^{100} by means of Euler's theorem.

   (d) If m and n are relatively prime positive integers, prove that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \mod mn.$$