# Homework #4

1. What are the possible values of the last digit of $4^m$ for each integer $m \geq 1$.

2. Show that a 3-digit positive integer $abc$ is divisible by 7 if and only if $2a+3b+c$ is divisible by 7.

3. Let $p$ be a prime number.

   (a) Show that if $k$ is an integer such that $1 \leq k \leq p-1$, then

   $$\binom{p}{k} \equiv 0 \mod p.$$

   Here $\binom{p}{k}$ is the binomial coefficient.

   (b) If $a$ and $b$ are integers, show that $(a+b)^p \equiv a^p + b^p \mod p$.

   (c) Show that if $a$ and $b$ are integers such that $a^p \equiv b^p \mod p$, then $a^p \equiv b^p \mod p^2$.

4. What are the natural numbers $n$ for which $1! + ... + n!$ is a perfect square? How about $(1!)^2 + ... + (n!)^2$? [Hint: consider $\mod 5$].

5. For each of the congruences $ax + b \equiv 0 \mod n$ below, find all its solutions mod $n$:

   (a) $3x + 7 \equiv 0 \mod 11$,

   (b) $4x + 22 \equiv 0 \mod 12$,

   (c) $382x + 121 \equiv 0 \mod 563$,

   (d) $55x - 3000 \equiv 0 \mod 121$.

6. Find all the numbers between 3000 and 3990 that are $\equiv 1 \mod 5, \equiv 2 \mod 9$ and $\equiv 3 \mod 11$.

7. Given any positive integer $k$, prove that there are $k$ consecutive integers each divisible by a square $> 1$. [Hint: Chinese Remainder Theorem!]