Homework #1

1. Establish the Bernoulli inequality: if $1 + a > 0$ then
$$(1 + a)^n \geq 1 + na$$
for all $n \geq 1$

2. Prove that $n! > n^2$ for every inter $n \geq 4$, whereas $n! > n^3$ for every integer $n \geq 6$.

3. Suppose that the sequence $\{a_n\}$ of numbers defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and
$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$
for all $n \geq 4$. Show that $a_n < 2^n$.

4. Use the Division Algorithm to establish the following:
   (a) The square of any integer is either of the form $3k$ or $3k + 1$.
   (b) The fourth power of any integer is either of the form $5k$ or $5k + 1$.
   (c) $3a^2 - 1$ is never a perfect square for any integer a.

5. For any integer a prove that
   (a) $\gcd(2a + 1, 9a + 4) = 1$
   (b) $\gcd(5a + 2, 7a + 3) = 1$

6. (a) Prove that the product of any three consecutive integers is divisible by 6.
   (b) Prove that the expression $(3n)!/(3!)^n$ is always an integer for every $n \geq 1$.

7. Confirm the following properties of the greatest common divisor:
   (a) If $\gcd(a, b) = \gcd(a, c) = 1$, then $\gcd(a, bc) = 1$. [Note: use the hypothesis to show that 1 can be written as $a\alpha + (bc)\beta$ for some integers α and β.]
   (b) If $\gcd(a, b) = 1$ and $c \mid (a + b)$ then $\gcd(a, c) = \gcd(b, c) = 1$.
   (c) If $\gcd(a, b) = 1$, $d \mid ac$ and $d \mid bc$, then $d \mid c$.

8. Prove that if $d \mid n$, then $2^d - 1$ divides $2^n - 1$. [Hint: use the identity
$$x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + ... + x + 1).]$$