Section 4.1 Solutions

1.1 a) False. The linear operator $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ on the 2-dimensional space $\mathbb{R}^2$ has only one eigenvalue, $\lambda = 2$, with multiplicity 2.

b) True. Suppose v is an eigenvector to the eigenvalue λ and matrix A. This means

$$(A - \lambda I)\mathrm{v} = 0$$

but then

$$\alpha(A - \lambda I)\mathrm{v} = \alpha 0$$

and therefore

$$(A - \lambda I)(\alpha \mathrm{v}) = 0$$

meaning that αv is an eigenvector to λ as well. Therefore if v is an eigenvector then αv is an eigenvector for any scalar α.

c) True. For example, the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues.

g) False. Given the matrix $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ it has eigenvalues

$$\begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} \text{ for } \lambda = 2$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } \lambda = -1.$$

But

$$\begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 2 \end{bmatrix}$$

is not an eigenvector for either 2 or -1.

h) True. Suppose that v and w are both eigenvectors to the same eigenvalue λ and matrix A. Then

$$(A - \lambda I)\mathrm{v} = 0 \text{ and } (A - \lambda I)\mathrm{w} = 0,$$

and since $A - \lambda I$ is a linear transformation,

$$(A - \lambda I)(\mathrm{v} + \mathrm{w}) = (A - \lambda I)\mathrm{v} + (A - \lambda I)\mathrm{w} = 0 + 0 = 0.$$

Therefore v + w is an eigenvector to λ.

1.2 (a) $p(\lambda) = \lambda^2 - \lambda - 2$

$$\lambda = 2, \quad \mathrm{v} = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

$$\lambda = -1, \quad \mathrm{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) $p(\lambda) = \lambda^2 - 6\lambda + 9$

$$\lambda = 3, \quad \mathrm{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) $p(\lambda) = (1-\lambda)(2+\lambda)^2$

$$\lambda = 1, \qquad \mathrm{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda = -2, \quad \mathrm{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

1.5 Suppose that A is an $n \times n$ triangular matrix with diagonal entires $a_1, a_2, ..., a_n$. Then $A - \lambda I$ is still a triangular matrix with diagonal entries

$$a_1 - \lambda, a_2 - \lambda, ..., a_n - \lambda.$$

Therefore the characteristic polynomial, $p(\lambda)$, of $A - \lambda I$ is just

$$p(\lambda) = \det(A - \lambda I) = (a_1 - \lambda)(a_2 - \lambda)\cdots(a_n - \lambda)$$

since it is still just the product of the diagonal entries. But the eigenvalues of A are just the roots of the characteristic polynomial which are just

$$a_1, a_2, ..., a_n,$$

the diagonal entries of A.