Review for Exam II: Chapters 2, 3, 4 of LADW.

def A **linear system** is a system of m linear equations in n unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

this can be written in matrix form as $\boxed{Ax = b}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

this is the coefficient matrix

it can also be expressed as an augmented matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & a_{mn} & b_m \end{array}\right]$$

To solve the system we perform row operations:

1. interchange rows $R_i \longleftrightarrow R_j$
2. mult a row by scalar $R_i \longmapsto \alpha R_i$
3. row replacement $R_i \longmapsto R_i + \alpha R_j$

Goal: to get the augmented matrix in echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1' \\ 0 & 1 & \cdots & 0 & b_2' \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & b_m' \end{array}\right]$$

get the upper left entry to equal 1, and everything below it equal 0

these non-zero entries with zeros ~~[illegible]~~ below them are called <u>pivots</u>.

<u>Exercise 1</u>

Be able to put a matrix

$$\begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

in echelon form.

<u>Exercise 2</u>

Be able to solve a linear system

$$x_1 + 2x_2 = 1$$
$$x_3 + 5x_4 = 2$$
$$x_5 = 3$$

<u>def</u> A system is called <u>inconsistent</u> if the reduced augmented matrix has a row of the form

$$[0 \;\cdots\; 0 \mid b]$$

where $b \neq 0$.

* A system has a unique solution iff it has a pivot in every column and every row.

<u>Exercise 3</u> Give an example of a system which is inconsistent. Give an example of a system which does not have unique solutions.

Important propositions from this section:

* proposition 3.1 (p. 48)
* proposition 3.2 (p. 48)
* proposition 3.4 (p. 49)
* proposition 3.6 (p. 50)

Exercise 4

Use the propositions above to determine if

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\3\\6\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}$$

is a basis for $\mathbb{R}^3$

Exercise 5

Use the propositions above to determine whether

$$\begin{bmatrix}1 & -1 & 2 & -1\\2 & 2 & 3 & 2\\3 & 1 & 0 & 1\\4 & 0 & 1 & 1\end{bmatrix}$$

is invertible.

* An augmented matrix can also be used to find an inverse matrix:

$$\left[\begin{array}{c|ccc} A & 1 & 0 & \\ & 0 & 1 & \\ & & & \ddots \\ & & & & 1\end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & & & \\ 0 & 1 & & & A^{-1} \\ & & \ddots & & \\ & & & 1 & \end{array}\right]$$

Exercise 6

Use the augmented matrix method to find the inverse of

$$A = \begin{bmatrix}1 & 0 & 2\\2 & 6 & 0\\0 & 2 & 0\end{bmatrix}$$

Recall : for a linear transformation $A : \mathbb{V} \longrightarrow \mathbb{W}$

$$\ker(A) = \{ v \in \mathbb{V} : Av = \vec{0} \} = \text{null}(A)$$

$$\text{range}(A) = \{ w \in \mathbb{W} : Av = w \text{ for some } v \in \mathbb{V} \} = \text{col}(A)$$

def the <u>fundamental ~~of~~ subspaces</u> for A are

$$\ker(A),\ \text{range}(A),\ \ker(A^T),\ \text{range}(A^T)$$

<u>Exercise 7</u>

Be able to compute the fundamental subspaces for

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Important theorems from this section:

* Theorem 7.1 (p. 63)
* Theorem 7.2 (p. 64)

<u>def</u> The <u>dimension</u> of a vector space is the # of vectors in its basis.

<u>def</u> The <u>rank</u> of a linear transformation is the dimension of the range.

<u>Exercise 8</u>

What is the rank of A from exercise 7?

* $\dim(\mathbb{R}^n) = n$
* $\dim(M_{m \times n}) = m \times n$
* $\dim(\mathbb{P}^n) = n+1$

def For a vector space $\mathbb{V}$ with bases $\mathcal{A}$ and $\mathcal{B}$,

$$[I]_{\mathcal{B}\mathcal{A}} : \mathbb{V} \longrightarrow \mathbb{V}$$

transforms v in terms of $\mathcal{A}$ to v in terms of $\mathcal{B}$, this is called a change of coordinate matrix.

Exercise 9

Write the change of coordinate matrix sending

$$\mathcal{A} = \{1, 1+t, 1+t+t^2\}$$

to

$$\mathcal{B} = \{2, t, 3t^2\}$$

in $\mathbb{P}^2$

If $T : \mathbb{V} \longrightarrow \mathbb{W}$ is a linear transformation and $A_{\mathbb{V}}, S_{\mathbb{V}}, A_{\mathbb{W}}, S_{\mathbb{W}}$ are bases for $\mathbb{V}$ and $\mathbb{W}$ then we can construct change of coordinate matrices.

$$[T]_{A_{\mathbb{W}} S_{\mathbb{V}}} = [I]_{A_{\mathbb{W}} S_{\mathbb{W}}} [T]_{S_{\mathbb{W}} A_{\mathbb{V}}} [I]_{A_{\mathbb{V}} S_{\mathbb{V}}}$$

Exercise 10

Using the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, write the linear transformation that sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ as $[T]_{\mathcal{A}\mathcal{B}}$

where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

__def__ for a system of vectors $v_1, v_2, \dots, v_n$, their __determinant__

$$\det(v_1, v_2, \dots, v_n)$$

is the volume of the fundamental parallelipiped spanned by $v_1, \dots, v_n$.

* if A is a matrix with columns $v_1, \dots, v_n$

$$\det(A) = \det(v_1, v_2, \dots, v_n)$$

__properties of determinants__

① mult. linearity: $\det(\alpha v_1, v_2, \dots, v_n) = \alpha \det(v_1, \dots, v_n)$

② additive linearity:

$$\det(v_1, \dots, v_i + u_i, \dots, v_n) = \det(v_1, \dots, v_i, \dots, v_n) + \det(v_1, \dots, u_i, \dots, v_n)$$

③ ~~column replacement~~: antisymmetry:

$$\det(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\det(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

④ ~~antisymmetry~~ Column replacement:

$$\det(v_1, \dots, v_i + \alpha v_j, \dots, v_n) = \det(v_1, \dots, v_i, \dots, v_n)$$

⑤ Normalization:

$$\det(e_1, e_2, \dots, e_n) = 1 = \det(I)$$

standard basis vectors.

⑥ $\det(A) = \det(A^T)$

⑦ determinant of a triangular matrix is just the product of the diagonal entries.

Exercise 11

Using properties of determinants, compute

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & -1 & 2 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$

Exercise 12

Compare det(A) and det(B) for

$$A = \begin{bmatrix} 1 & x \\ 1 & y \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3x+1 \\ 2 & 3y+1 \end{bmatrix}$$

important propositions from this section

* proposition 3.1 (p. 79)
* proposition 3.3 (p. 82)
* theorem 3.5 (p. 82)

Exercise 13

Let A be a square matrix, show that

$$\det(A) = \begin{vmatrix} I & * \\ 0 & A \end{vmatrix} = \begin{vmatrix} A & * \\ 0 & I \end{vmatrix} = \begin{vmatrix} I & 0 \\ * & A \end{vmatrix} = \begin{vmatrix} A & 0 \\ * & I \end{vmatrix}$$

The cofactor expansion of a determinant is given by:

$$\det(A) = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \text{entry in row } j, \\ \text{column } k \text{ of } A}}{a_{j,k}} \cdot (-1)^{j+k} \cdot \det(\underbrace{A_{j,k}}_{\substack{\uparrow \\ \text{matrix obtained from crossing out} \\ \text{row } j \text{ and column } k \text{ of } A.}})$$

or, equivalently:

$$\det(A) = \sum_{j=1}^{n} a_{j,k} \cdot (-1)^{j+k} \det(A_{j,k})$$

__exercise 14__

Use cofactor expansion to compute

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

__def__ A scalar λ is called an __eigenvalue__ of $A : \mathbb{V} \to \mathbb{V}$ if there is some vector $v \in \mathbb{V}$ such that

$$Av = \lambda v$$

In this case v is called an __eigenvector__ for A and λ.

__exercise 15__

Compute the __characteristic polynomial__, eigenvalues and eigenvectors for

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$p(\lambda) = \det(A - \lambda I)$

* eigenvalues are just roots of the characteristic poly., $p(\lambda)$.

* * * And don't forget to review all HW problems * * *