Review for Exam 1: Chapter 1 of LADW

def A <u>vector space</u>, $\mathbb{V}$, is a collection of objects called vectors, denoted v, and two operations, $+$ and $\times$.

$+$: vector addition, $v+w$

$\times$: scalar multiplication, $\alpha \cdot v$

* vector spaces have 8 axioms. what are they?

① $v+w = w+v$

② $v+(w+u) = (v+w)+u$

③ $v+\vec{0} = v$

④ $\forall v \in \mathbb{V}, \exists w \in \mathbb{V}$, s.t. $v+w=\vec{0}$

⑤ $1 \cdot v = v$

⑥ $(\alpha\beta)v = \alpha(\beta v)$

⑦ $\alpha(u+v) = \alpha u + \alpha v$

⑧ $(\alpha+\beta)u = \alpha u + \beta u$

<u>claim 1:</u> $\vec{0}$ is unique.

<u>claim 2:</u> inverses are unique.

* $\mathbb{V}$ a vector space
* $\{v_1, v_2, \dots, v_p\}$ a collection of vectors.

def A <u>linear combination</u> of vectors is a sum of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p = \sum_{i=1}^{p} \alpha_i v_i$$

__def__: if $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p$ is a linear combination, then $\alpha_1, \alpha_2, \dots, \alpha_p$ are called the __coordinates__ of v.

__def__: A collection $\{v_1, \dots, v_n\}$ in $\mathbb{V}$ is called a __generating system__ if every $v \in \mathbb{V}$ can be written as a linear combination of $\{v_1, \dots, v_n\}$.

__claim 3__:
$\{1, t, t^2\}$ is a generating system for $\mathbb{P}^2$.

__claim 4__:
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is __not__ a generating system for $\mathbb{R}^3$.

__def__ A linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called __trivial__ if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

* A trivial linear combination always equals the zero vector.

$$0v_1 + 0v_2 + \dots + 0v_n = \vec{0} + \vec{0} + \dots + \vec{0} = \vec{0}.$$

__def__ A collection $\{v_1, \dots, v_n\}$ in $\mathbb{V}$ is called __linearly independent__ if only the trivial linear combination equals $\vec{0}$.

__claim 5__
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is not linearly independent

Claim 6

$\{1, 1-t, t^2\}$ is linearly independent.

Def A collection $\{v_1, \dots, v_n\}$ in $\mathbb{V}$ is called <u>linearly dependent</u> if it is not linearly independent.

* this means $\vec{0} = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ where $\sum_{i=1}^{n} |\alpha_i| \neq 0$.

Claim 7

In $M_{2\times 2}$ show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is linearly dependent.

Def A collection $\{v_1, \dots, v_n\}$ in $\mathbb{V}$ is called a <u>basis</u> if it is a linearly independent generating set.

Claim 8

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

* to show something is a basis:

1. show it is linearly independent.
2. show it is a generating system.

<u>How to write a vector</u>

* $\mathbb{V}$ a vector space.

* $\{v_1, \dots, v_n\}$ a basis for $\mathbb{V}$.

$\rightarrow$ for any $v \in \mathbb{V}$, $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

* a vector in $\mathbb{V}$ is uniquely identified by its column of scalars

def Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A transformation

$$T : \mathbb{V} \longrightarrow \mathbb{W}$$

is called linear if

① $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in \mathbb{V}$

② $T(\alpha v) = \alpha T(v) \quad \forall\, v \in \mathbb{V}$, α scalar

Claim 9

The transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that reflects vectors over the y-axis is a linear transformation.

Claim 10

For any linear transformation $T : \mathbb{V} \longrightarrow \mathbb{W}$,

$$T(\vec{0}) = \vec{0}$$

* Every linear transformation $T : \mathbb{V} \longrightarrow \mathbb{W}$ can be expressed as a matrix, $[T]$, where

$$[T] = \begin{bmatrix} T(v_1) & T(v_2) & \cdots & T(v_n) \end{bmatrix}$$

where $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{V}$, and $T(v) = [T]v$.

Claim 11

The linear transformation $T : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ defined by $T(p(t)) = p(t) + p'(t)$ has the associated matrix

$$[T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

claim 12

for any two vector spaces $\mathbb{V}$ and $\mathbb{W}$, the collection of all linear transformations from $\mathbb{V}$ to $\mathbb{W}$, denoted $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is a vector space.

* for linear transformations $T_1 : \mathbb{V} \longrightarrow \mathbb{W}$ and $T_2 : \mathbb{U} \longrightarrow \mathbb{V}$, the composition of T_1 and T_2 is given by

$$\begin{aligned} T_1 T_2 : \mathbb{U} &\longrightarrow \mathbb{W} \\ u &\longmapsto T_1(T_2(u)) \end{aligned}$$

and $[T_1 T_2] = [T_1][T_2]$.

* dimension is important in matrix multiplication,

$$(M_{m \times n}) \cdot (M_{n \times \ell}) = (M_{m \times \ell})$$

claim 13

for matrices A and B

$$(AB)^T = B^T A^T$$

claim 14

Let $T_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be reflection over the y-axis, and $T_x : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection over the x-axis. Then

$$[T_x T_y] = [T_x][T_y] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

def the identity transformation, I, is a map

$$I : \mathbb{V} \longrightarrow \mathbb{V}$$
$$v \longmapsto v$$

def A linear transformation $T : \mathbb{V} \longrightarrow \mathbb{W}$ is

1. <u>left invertible</u> if $AT = I$ for $A : \mathbb{W} \longrightarrow \mathbb{V}$.
2. <u>right invertible</u> if $TB = I$ for $B : \mathbb{W} \longrightarrow \mathbb{V}$.
3. <u>invertible</u> if $AT = I = TB$ and $A = B$ is unique.

claim 15
if A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

claim 16
The linear trans. with associated matrix
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
is not invertible

claim 17
The linear transformation with associated matrix
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
is invertible.

* we call a matrix invertible if its associated linear transformation is invertible.

* only square matrices are invertible.

def An invertible linear transformation is called an <u>isomorphism</u>.

* A linear transformation is invertible iff it sends a basis to a basis.

Claim 18

The linear transformation $T: \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ from claim 11 is invertible.

def A subspace of a vector space $\mathbb{V}$ is a nonempty set $\mathbb{V}_0 \subseteq \mathbb{V}$ which is closed under vector addition and scalar multiplication.

Claim 19

For a linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$, both $\ker(T)$ and range (T) are subspaces. Recall:

$$\ker(T) = \{v \in \mathbb{V} : T(v) = \vec{0}\}$$

and

$$\text{range}(T) = \{w \in \mathbb{W} : T(v) = w \text{ for some } w \in \mathbb{W}\}$$