Section 3.3 Solutions

3.1 Suppose that $A \in M_{n\times n}$ is the matrix with columns $a_1, ..., a_n$. Then

$$\det(A) = \det(a_1, ..., a_n).$$

We know that $5A$ corresponds to multiplying each entry of A by 5, that is, the columns of $5A$ are $5a_1, ..., 5a_n$. Therefore,

$$\det(5A) = \det(5a_1, ..., 5a_n) = 5^n \det(a_1, ..., a_n) = 5^n \det(A).$$

3.2 (a) Suppose that

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{bmatrix}.$$

Then

$$2 \cdot 3 \cdot 5 \cdot \det(A) = \det(B).$$

(b) Suppose that

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{bmatrix}.$$

Then we know that

$$\begin{vmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{vmatrix} = \begin{vmatrix} 3a_1 & 4a_2 & 5a_3 \\ 3b_1 & 4b_2 & 5b_3 \\ 3c_1 & 4c_2 & 5c_3 \end{vmatrix} + \begin{vmatrix} 3a_1 & 5a_1 & 5a_3 \\ 3b_1 & 5b_1 & 5b_3 \\ 3c_1 & 5c_1 & 5c_3 \end{vmatrix} = \begin{vmatrix} 3a_1 & 4a_2 & 5a_3 \\ 3b_1 & 4b_2 & 5b_3 \\ 3c_1 & 4c_2 & 5c_3 \end{vmatrix}$$

and therefore

$$3 \cdot 4 \cdot 5 \cdot \det(A) = \det(B).$$

3.3 (a) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = -12$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -12 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 7 & -6 & 0 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 4 & 19 & -43 \\ 2 & 3 & 19 & -38 \end{vmatrix} = 19 \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 2 & 3 & 1 & 5 \end{vmatrix} = 95$

(d) $\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ x & y - x \end{vmatrix} = y - x$

3.4 Suppose that $A \in M_{n\times n}$ is a skew-symmetric matrix, that is, $A^T = -A$, and $n = 2k+1$. Let $a_1, ..., a_{2k+1}$ be the columns of A. Then we have

$$\begin{aligned} \det(A^T) &= \det(-A) \\ &= \det(-a_1, ..., -a_{2k+1}) \\ &= (-1)^{2k+1} \det(a_1, ..., a_{2k+1}) \\ &= -\det(A) \\ &= -\det(A^T), \end{aligned}$$

since $\det(A) = \det(A^T)$. Therefore, $\det(A^T) = \det(A) = 0$. This is not the case when n is even, since for example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

has

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

but $\det(A) = 1$.

3.5 Suppose that $A \in M_{n\times n}$ is nilpotent, that is, $A^k = 0$ for some positive integer k. In order to prove that $\det(A) = 0$, we will show that A is not invertible. For the sake of contradiction, suppose that A is invertible, so there is some A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Then,

$$0 = 0 \cdot (A^{-1})^k = A^k(A^{-1})^k = (AA^{-1})^k = I^k = I,$$

which is a contradiction, since clearly $0 \neq I$. Therefore, A is not invertible, and therefore $\det(A) = 0$ (see Proposition 3.3).

3.7 Suppose that $Q \in M_{n\times n}$ is orthogonal, that is, $Q^TQ = I$. Since

$$\det(Q) = \det(Q^T),$$

and the determinant is multiplicative, this means

$$1 = \det(I) = \det(Q^TQ) = \det(Q^T)\det(Q) = \det(Q)^2.$$

Therefore, since $\det(Q)^2 = 1$ it follow that $\det(Q) = \pm 1$.

3.8

$$\begin{aligned}
\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & y-x & y^2-x^2 \\ 1 & z-x & z^2-x^2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 1 & y-x & (y-x)(y+x) \\ 1 & z-x & (z-x)(z+x) \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 1 & y-x & 0 \\ 1 & z-x & (z-x)(z+x)-(z-x)(y+x) \end{vmatrix} \\
&= (y-x)[(z-x)(z+x)-(z-x)(y+x)] \\
&= (z-x)(y-x)[(z+x)-(y+x)] \\
&= (z-x)(y-x)(z-y)
\end{aligned}$$