Section 2.8 Solutions

8.1 (a) True. By definition a change of coordinate matrix for a vector space V is a matrix $[I]_{\mathcal{BA}}$ which takes any vector $v \in V$ written in basis $\mathcal{A}$ and spits out that same vector written in terms of the basis $\mathcal{B}$. But we know that $\mathcal{A}$ and $\mathcal{B}$ contain the same number of vectors (see Proposition 3.3), say n many. This means, whether v is written in terms of $\mathcal{A}$ or $\mathcal{B}$, it has the same number of coordinates. Therefore $[I]_{\mathcal{BA}}$ sends a vector with n coordinates to a vector with n coordinates, and hence it must be square.

(b) True. Let V be a vector space with bases $\mathcal{A}$ and $\mathcal{B}$. Suppose that $[I]_{\mathcal{BA}}$ is a change of coordinate matrix from $\mathcal{A}$ to $\mathcal{B}$ for a vector space V. Then the inverse of $[I]_{\mathcal{BA}}$ is just the change of coordinate matrix $[I]_{\mathcal{AB}}$ from $\mathcal{B}$ to $\mathcal{A}$.

8.2 (a) To show that the given system of vectors is a basis in $\mathbb{F}^4$, we simply example with corresponding column matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

which in echelon form is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there is a pivot in every column and every row, we know the system of vectors form a basis.

(b) If we let

$$\mathcal{A} = \{(1,2,1,1)^T, (0,1,3,1)^T, (0,3,2,0)^T, (0,1,0,0)^T\}$$

then

$$[I]_{\mathcal{SA}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

8.3 Consider the vector space $\mathbb{P}^1$ and consider the bases

$$\mathcal{A} = \{1, 1+t\}$$

and

$$\mathcal{B} = \{1-t, 2t\}.$$

Then $[I]_{\mathcal{B}\mathcal{A}}$ will be a 2×2 matrix whose first column expresses 1 as a coordinate vector in terms of $\mathcal{B}$ and its second column expresses $1+t$ as a coordinate vector in terms of $\mathcal{B}$. Since

$$1 = (1)(1-t) + (1/2)(2t) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}_{\mathcal{B}} \text{ and } 1+t = (1)(1-t) + (1)(2t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

we have

$$[I]_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

8.4 Let T be a linear transforamtion on the space $\mathbb{F}^2$ defined in the standard coordinates by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ x-2y \end{bmatrix},$$

and define the bases

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

for $\mathbb{F}^2$. This question is asking us to find $[I]_{\mathcal{S}\mathcal{S}}$ and $[I]_{\mathcal{A}\mathcal{S}}$. The matrix $[I]_{\mathcal{S}\mathcal{S}}$ is just the usual matrix for a linear transformation that we're used to, that is

$$[I]_{\mathcal{S}\mathcal{S}} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix},$$

where the columns are just the image of each of the standard bases elements under the transformation T. To find $[I]_{\mathcal{A}\mathcal{S}}$, we need to find the image of each of the elements of $\mathcal{S}$ under T, and express them in terms of $\mathcal{A}$. That is,

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{S}} \right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{S}} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{\mathcal{A}},$$

and

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{S}} \right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\mathcal{S}} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}_{\mathcal{A}},$$

which means

$$[I]_{\mathcal{A}\mathcal{S}} = \begin{bmatrix} 5 & 4 \\ -2 & -3 \end{bmatrix}.$$