Section 2.7 Solutions

7.1 (a) False. The linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ with associated matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank 2 but has no non-zero columns.

(b) True. We know from Theorem 7.2 that

$$\dim(\ker(A)) + \dim(\text{range}(A)) = \dim(\ker(A)) + \text{rank}(A) = n$$

So suppose that an $m \times n$ matrix has rank 0. This means

$$\dim(\ker(A)) = n.$$

But we also know the dimension of the kernel is equal to the number of free variables, and

$$\{ \text{ number of free variables } A\vec{x} = \vec{0}\} = n - \{ \text{ the number of pivots in } A_e\}.$$

This means A_e has no pivots, which is only possible if A is the all zeroes matrix.

(c) True. Elementary row operations will not change the solutions of the homogeneous equation, $A\vec{x} = \vec{0}$, therefore they do not change the dimension of $\ker(A)$. But recall from Theorem 7.2 that

$$\dim(\ker(A)) + \text{rank}(A) = n$$

we conclude that elementary row operations do not change the rank of A.

(d) False. Elementary column operations of A are just elementary row operations of A^T, but by the rank theorem

$$\text{rank}(A) = \text{rank}(A^T).$$

Therefore elementary column operations *necessarily* preserve the rank of A.

(e) True. Suppose $\text{rank}(A) = k$ and A has m linearly independent columns. Each of the linearly independent columns must contain a pivot by Proposition 3.1. But from the definition of rank, we know that there are k columns that contain pivots. Therefore $m \leq k$.

(f) True. Suppose that A is a matrix with rank k. Recall that

$$\text{rank}(A) = k = \text{rank}(A^T).$$

If A has m linearly independent rows, then A^T has m linearly independent columns. But then from problem 5 we know that $m \leq k$.

(g) True. The rank of a matrix is the number of pivot columns in the echelon form of the matrix, this is necessarily less than or equal to the number of columns in the original matrix.

(h) True. If an $n \times n$ matrix, A, has rank n, this means A_e has a pivot in each column. This means the columns of A were linearly independent. This also means A_e has a pivot in each row, and hence the columns of A were a generating system. Therefore the columns of A form a basis, and if the columns of a matrix form a basis, this associated linear transformation is invertible.

7.3 (a) If

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

then

$$A_e = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A_e^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore rank(A)=rank(A^T)=2, and the bases for the range of A and A^T can be read directly from the original matrices by finding the pivot columns of A_e and A_e^T. Solving the homogeneous equation $A\vec{x} = \vec{0}$, we obtain

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and hence ker(A) has the basis

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Similarly, solving the homogeneous equation $A^T\vec{x} = \vec{0}$, we obtain

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and hence ker(A^T) has the basis

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(b) If

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

then

$$A_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A_e^T = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since both have three pivots the rank of A is 3, and the bases for the range of A can be read directly from the original matrices by finding the pivot columns of A_e and A_e^T. Again by solving the inhomogeneous equation $A\vec{x} = \vec{0}$ we get

$$\vec{x} = x_4 \begin{bmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{6} \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore $\ker(A)$ has a basis

$$\begin{bmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{6} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly, solving $A^T\vec{x} = \vec{0}$ we get

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

and therefore $\ker(A^T)$ has a basis

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

7.4

7.8 (a) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) This is impossible. If the columns space of a matrix, A, is spanned by $(1,1,1)^T$, then we already know that A has precisely 3 rows. Also, if the nullspace is spanned by $(1,2,3)^T$ then the matrix must have three columns (in order for matrix multiplication to be well defined). But now A is a 3×3 matrix with rank 1, and therefore the nullspace must have dimension 2.

(c) This is impossible. If the column space of a matrix, A, is $\mathbb{R}^4$, this means $\text{rank}(A) = 4$. But if the row space of A is $\mathbb{R}^3$, then $\text{rank}(A^T) = \mathbb{R}^3$, but we know that

$$\text{rank}(A) = \text{rank}(A^T)$$

7.9 No, for example, consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Clearly A and B are not equal. But since the are already in echelon form we can see that they both have rank 2. This means that

$$\dim(\ker(A)) = \dim(\ker(A^T)) = \dim(\ker(B)) = \dim(\ker(B^T)) = 0$$

and so

$$\ker(A) = \ker(A^T) = \ker(B) = \ker(B^T) = \{\vec{0}\}.$$

On the other hand,

$$\text{range}(A) = \text{range}(A^T) = \text{range}(B) = \text{range}(B^T) = \mathbb{R}^2.$$