SECTION 2.5 SOLUTIONS

5.1 (a) True. By proposition 5.1 the vector space must be finite dimension, and therefore by definition it has a finite basis.

(b) False. An infinite vector space can have an infinite basis.

(c) False. The vector space $\mathbb{R}^2$ has the two bases,

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

and

$$\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}.$$

(d) True. This is a proposition we proved in class.

(e) False. The space of $M_{m \times n}$ matrices has dimension $m \times n$. For example, we know that a basis for the space $M_{2\times 1}$ is

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$$

giving the space dimension 2.

(f) False. The basis for $\mathbb{P}^n$ contains $n + 1$ vectors.

(g) False. The vectors

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\0\\1\end{bmatrix}$$

span $\mathbb{R}^3$, but the all 1's vector can clearly be written in two different ways.

(h) True. Suppose that V is a finite dimensional space with subspace W. From proposition 5.4 and from the definition of subspace, we know that the basis for W is a linearly independent collection of vectors in V can be completed to a basis. But any basis for V is finite. Therefore the basis for W must have been finite to begin with.

(i) True. The 0 dimensional space is $\{\vec{0}\}$ and the n dimensional space is the space itself.

5.2 Suppose that V is a vector space having dimension n, and suppose that $\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ is a system of vectors in V. Let

$$A = \begin{bmatrix}\mathrm{v}_1 & \mathrm{v}_2 & \cdots & \mathrm{v}_n\end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & & v_{2n} \\ \vdots & & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$$

be the associated matrix whose columns are the v_i. Since A is a square matrix it has a pivot in every column if and only if it has a pivot in every row. Therefore the system $\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ is linearly independent if and only if it is a generating system.

5.3 Suppose that $v_1, v_2, ..., v_n$ is a linearly independent system of vectors in a vector space V. If this system is a basis, then by definition $\dim(V) = n$. Suppose that $\dim(V) = n$. Then a basis can have at most n vectors, and by proposition 5.4 any linearly independent system in V can be completed to a basis. Since $v_1, v_2, ..., v_n$ already contains n vectors, we can conclude that this is a basis.

5.6 (a) The column matrix,

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 1 & 0 & 50 \\ 5 & 0 & -921 \\ -3 & 0 & 0 \end{bmatrix}$$

can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and since there is a pivot in each column the system is linearly independent.

(b) Adding the vectors

$$(0, 0, 0, 1, 0)^T \text{ and } (0, 0, 0, 0, 1)^T$$

will give a pivot in each column and each row, hence yielding a basis.