# Section 2.3 Solutions

3.1 We would like to know, for which values $b$ does the system

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ b \end{bmatrix}$$

have a solution (i.e. for which $b$ is it "consistent")? Recall, the system is consistent if and only if its echelon form does not have a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array}\right]$$

where $b \neq 0$. Building the augmented matrix we have

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & b \end{array}\right]$$

which can be reduced to echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & b-2 \end{array}\right].$$

Therefore the system is consistent if and only if $b = 2$. Using this value of $b = 2$ we have the echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

giving us the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

3.2 The vectors are linearly independent if and only if the echelon form of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has a pivot in every column. By performing row operations, we see that this matrix has echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which does not have a pivot in the fourth column. Therefore the set of vectors is not linearly independent. They also fail to be a spanning system for both $\mathbb{R}^4$ since there isn't a pivot in every row. And we can ignore $\mathbb{C}^4$ for now.

3.3 a) The matrix associated to this system,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has a pivot in every row and column and is hence a basis.

b) The matrix associated to this system,

$$\begin{bmatrix} -1 & -3 & 2 \\ 3 & 1 & 10 \\ 2 & 3 & 2 \end{bmatrix}$$

has echelon form

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

which does not have a pivot in every row and column and hence is not a basis.

c) The matrix associated to this system,

$$\begin{bmatrix} 67 & \pi & 3 \\ 13 & -\frac{196}{25} & 0 \\ -47 & 0 & 0 \end{bmatrix}$$

has reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which has a pivot in every row and column and is hence a basis.

3.4 The vector space $\mathbb{P}^3$ has the basis $\{1, t, t^2, t^3\}$. In this basis, the polynomials given correspond to the column vectors

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since the associated matrix will have 4 rows and 3 columns, there is no way it can have a pivot in every row. Therefore the set of vectors is not a spanning system.

3.5 Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ are five vectors in $\mathbb{F}^4$. Their associated matrix

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}$$

will have 4 rows and 5 columns, so there is no way to have a pivot in every column. Therefore the set of vectors cannot be linearly independent.

3.6 Suppose that $A \in M_{n\times n}$ and the columns of $A$ are linearly independent. This means that the echelon form of $A$ has a pivot in every column, and since $A$ is square, the echelon form of $A$ also has a pivot in every row. This means that the *reduced* echelon form of $A$ is the identity matrix. Recalling that row operations are just linear transformations, this means that there is some series of linear transformations $E_1, E_2, E_3, ..., E_m$ corresponding to each of the row operations with

$$E = E_m \cdots E_2 E_1$$

(the composition of the $E_i$ linear transformations) such that $EA = I$. Therefore $E = A^{-1}$. Consequently,

$$E^2A^2 = A^{-1}A^{-1}AA = A^{-1}IA = A^{-1}A = I$$

and therefore $A^2$ can be reduced via the sequence of row operations

$$E_1, E_2, ..., E_m, E_1, E_2, ..., E_m$$

to obtain the reduced echelon form which has a pivot in every row and every column. Therefore the columns of $A^2$ are linearly independent.