Section 1.7 Solutions

7.1 Let X and Y be subspaces of a vector space V. We would like to show that $X \cap Y$ is a subspace of V. First, we recall that in set-builder notation

$$X \cap Y = \{w : w \in X \text{ and } w \in Y\}.$$

To show that $X \cap Y$ is nonempty, we note that since X and Y are vector spaces (since they are subspaces of V) we know that $\vec{0} \in X$ and $\vec{0} \in Y$ and hence $\vec{0} \in X \cap Y$. Since X is a subspace of V we know that $X \subseteq V$ and hence $(X \cap Y) \subseteq X \subseteq V$. To show that $X \cap Y$ is closed under vector addition, suppose that $v, w \in X \cap Y$. Then $v, w \in X$ and $v, w \in Y$, and since X and Y are subspaces, we know $v+w \in X$ and $v+w \in Y$. Therefore $v + w \in X \cap Y$. Finally, to show that $X \cap Y$ is closed under scalar multiplication, let $v \in X \cap Y$ and suppose α is a scalar. Then $v \in X$ and $v \in Y$, and since X and Y are subspaces $\alpha v \in X$ and $\alpha v \in Y$, consequently $\alpha v \in X \cap Y$.

7.2 Let X and Y be subspaces of a vector space V. We would like to show that $X + Y$ is a subspace of V. First, we recall that in set-builder notation

$$X + Y = \{w : w = x + y \text{ for some } x \in X \text{ and } y \in Y\}.$$

To show that $X + Y$ is nonempty, we note that since X and Y are vector spaces (since they are subspaces of V) we know that $\vec{0} \in X$ and $\vec{0} \in Y$ and hence $\vec{0} + \vec{0} = \vec{0} \in X + Y$. Let $w \in X + Y$, then $w = x + y$ where $x \in X \subseteq V$ and $y \in Y \subseteq V$, and since V is a vector space and hence closed under vector addition, $x + y \in V$. Therefore, $X + Y \subseteq V$. To show that $X \cap Y$ is closed under vector addition, suppose that $v, w \in X + Y$. Then $v = x + y$ and $w = x' + y'$, where $x, x' \in X$ and $y, y' \in Y$, and since X and Y are subspaces, we know $x + x' \in X$ and $y + y' \in Y$, and hence

$$v + w = x + y + x' + y' = x + x' + y + y' \in X + Y.$$

Finally, to show that $X + Y$ is closed under scalar multiplication, let $v \in X + Y$ and suppose α is a scalar. Then $v = x + y$ where $x \in X$ and $y \in Y$, and since X and Y are subspaces $\alpha x \in X$ and $\alpha y \in Y$, consequently

$$\alpha v = \alpha(x + y) = \alpha x + \alpha y \in X + Y.$$

7.3 Let X be a subspace of a vector space V, and let $v \in V$ and $v \notin X$. We want to show that if $x \in X$ then $x + v \notin X$. For the sake of contradiction, suppose that $x \in X$ and $x + v \in X$. Since X is a subspace and hence closed under vector addition and scalar multiplication, this means that

$$v = -(x) + (x + v) \in X$$

contradicting the assumption that $v \notin X$.

7.4 Let X and Y be subspaces of a vector space V. We would like to show that $X \cup Y$ is a subspace if and only if $X \subset Y$ or $Y \subset X$. We will begin with the backwards direction. If $X \subset Y$, then $X \cup Y = Y$ and hence $X \cup Y$ is a subspace of V. On the other hand, if $Y \subset X$, then $X \cup Y = X$ and hence $X \cup Y$ is a subspace of V.

The other direction can be proved by showing that the contrapositive is true, namely, we will show that if $X \not\subset Y$ and $Y \not\subset X$ then $X \cup Y$ is not a subspace of V. If $Y \not\subset X$ then there must be some vector $y \in V$ which is in in Y but not in X. Similarly, since $X \not\subset Y$ there must be some vector $x \in V$ which is in X but not in Y. Since $x \in X$ and $y \in Y$ we know that $x, y \in X \cup Y$. Now using exercise 7.3 above, we know that $x + y \notin X$ and $x + y \notin Y$. Therefore $x + y \notin X \cup Y$. Since $X \cup Y$ is not closed under vector addition, we know that it is not a subspace of V. Therefore we have show that if $X \cup Y$ is a subspace then $X \subset Y$ or $Y \subset X$