Section 1.6 Solutions

6.1 Suppose that $A : V \to W$ is an invertible linear transformation and $\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ is a basis in V. We would like to show that $A\mathrm{v}_1, A\mathrm{v}_2, ..., A\mathrm{v}_n$ is a basis in W, that is, we would like to show that $A\mathrm{v}_1, A\mathrm{v}_2, ..., A\mathrm{v}_n$ is a linearly independent generating set for W. To show that $A\mathrm{v}_1, A\mathrm{v}_2, ..., A\mathrm{v}_n$ is linearly independent, suppose that

$$\vec{0} = \alpha_1 A\mathrm{v}_1 + \alpha_2 A\mathrm{v}_2 + ... + \alpha_n A\mathrm{v}_n$$

for some scalars $\alpha_1, \alpha_2, ..., \alpha_n$. It would suffice to show that all of the α_i are necessarily equal to 0. Note that A^{-1} (and in fact any linear transformation) must send $\vec{0}$ to $\vec{0}$, since

$$\vec{0} = A^{-1}(\mathrm{w}) - A^{-1}(\mathrm{w}) = A^{-1}(\mathrm{w} - \mathrm{w}) = A^{-1}(\vec{0})$$

for any $\mathrm{w} \in W$. Therefore,

$$\begin{aligned}
\vec{0} &= A^{-1}(\vec{0}) \\
&= A^{-1}(\alpha_1 A\mathrm{v}_1 + \alpha_2 A\mathrm{v}_2 + ... + \alpha_n A\mathrm{v}_n) \\
&= A^{-1}(\alpha_1 A\mathrm{v}_1) + A^{-1}(\alpha_2 A\mathrm{v}_2) + ... + A^{-1}(\alpha_n A\mathrm{v}_n) \\
&= \alpha_1 A^{-1}(A\mathrm{v}_1) + \alpha_2 A^{-1}(A\mathrm{v}_2) + ... + \alpha_n A^{-1}(A\mathrm{v}_n) \\
&= \alpha_1 (A^{-1}A)(\mathrm{v}_1) + \alpha_2 (A^{-1}A)(\mathrm{v}_2) + ... + \alpha_n (A^{-1}A)(\mathrm{v}_n) \\
&= \alpha_1 (I)(\mathrm{v}_1) + \alpha_2 (I)(\mathrm{v}_2) + ... + \alpha_n (I)(\mathrm{v}_n) \\
&= \alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + ... + \alpha_n \mathrm{v}_n.
\end{aligned}$$

But since $\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ is a basis for V, we already know that it is linearly independent. Therefore the only way to express the $\vec{0}$ as a linear combination of v_i is via the trivial linear combination. Hence we may conclude that all of the α_i are equal to 0, which is what we needed to show.

To show that $A\mathrm{v}_1, A\mathrm{v}_2, ..., A\mathrm{v}_n$ is a generating set, we need to show that any $\mathrm{w} \in W$ can be written as linear combination of the $A\mathrm{v}_i$. Since A is invertible, we know that for every $\mathrm{w} \in W$ there is some $\mathrm{v} \in V$ such that $A(\mathrm{v}) = \mathrm{w}$. But since $\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n$ is a basis for V, we have

$$\mathrm{v} = \alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + ... + \alpha_n \mathrm{v}_n$$

for scalars $\alpha_1, \alpha_2, ..., \alpha_n$, and hence

$$\begin{aligned}
\mathrm{w} &= A(\mathrm{v}) \\
&= A(\alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + ... + \alpha_n \mathrm{v}_n) \\
&= \alpha_1 A\mathrm{v}_1 + \alpha_2 A\mathrm{v}_2 + ... + \alpha_n A\mathrm{v}_n),
\end{aligned}$$

which is what we wanted to show.

6.2 Suppose $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$. We would like to find every right inverse to B. Since $A \in M_{1\times 2}$, we know that $B \in M_{2\times 1}$ and $AA^{-1} \in M_{1\times 1}$. So we need to find all real numbers a and b for which

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

So any a, b satisfying the relation $a = 1 - b$ will satisfying this relation. Since there are infinitely many right inverses to A, we know that the right inverse is not unique. From this we can conclude that A is not invertible, since invertible matrixes have a unique right inverse.

6.3 Since $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T \in M_{3\times 1}$ we know that the left inverse $A^{-1} \in M_{1\times 3}$ and hence $A^{-1}A \in M_{1\times 1}$. Therefore we need to find numbers a, b, c which satisfy

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} a+2b+3c \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

Any numbers a, b, c which satisfying $a+2b+3c = 1$ will satisfy this equation. For example,

$$a = 1, b = 1, c = 0$$

or

$$a = 0, b = 0, c = 3.$$

There are infinitely many solutions.

6.4 From the preceding problem, we know that the column $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$ has infinitely many left inverses and therefore cannot have any right inverses, since it is not invertible.

6.5 Following the hint, suppose that $AB = \begin{pmatrix} 1 \end{pmatrix}$. Suppose that $A \in M_{1\times 2}$ and $B \in M_{2\times 1}$. Then

$$A = \begin{pmatrix} a & b \end{pmatrix}$$

and

$$B = \begin{pmatrix} c \\ d \end{pmatrix}$$

and hence

$$AB = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac+bd \end{pmatrix},$$

so we need $ac + bd = 1$. One possible solution is $c = d = \frac{1}{2}$ and $a = b = 1$. We already know from problem 6.2 above that A is not invertible. To show that B is not invertible, we need only notice that

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

Since B has two left inverses it cannot be invertible.

6.8 Let $\mathbf{0}$ denote the all zeroes matrix. Suppose that A is an $n \times n$. If $A = \mathbf{0}$ then $AB = \mathbf{0}$ for any matrix B and hence it is not invertible. Therefore we may assume that $A \neq \mathbf{0}$. Suppose that $A^2 = \mathbf{0}$, and for the sake of contradiction, suppose that A is invertible. Then there is some A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

However,

$$AAA^{-1} = (AA)A^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$$

while on the other hand

$$AAA^{-1} = A(AA^{-1}) = AI = A,$$

but this contradicts the fact that $A \neq \mathbf{0}$.

6.9 Suppose that $AB = \mathbf{0}$ for some non-zero matrix B. In what follows, we will show that A is not invertible. For the sake of contradiction, suppose that A is invertible, so there is some A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Then

$$A^{-1}AB = (A^{-1}A)B = IB = B$$

while on the other hand

$$A^{-1}AB = A^{-1}(AB) = A^{-1}\mathbf{0} = \mathbf{0}.$$

This contradicts the fact that B is non-zero. Therefore, A is not invertible.

6.10 We will let T_1 and T_2 be linear transformations from $\mathbb{F}$ to $\mathbb{F}^4$, defined by

$$T_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix}$$

and

$$T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + ax_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

We want write the matrices corresponding to T_1^{-1} and T_2^{-1}. Since T_1 just swaps x_2 and x_4, we would want T_1^{-1} to swap them back to their original places, that is it again swaps the second and fourth entries

$$T_1^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix}.$$

In this way,

$$T_1^{-1}T_1\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=T_1^{-1}\begin{pmatrix}x_1\\x_4\\x_3\\x_2\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}.$$

Similarly, since T_2 multiplies x_4 by a and adds it to x_2, we would want T_2^{-1} to subtract a times the fourth entry from the second entry, that is

$$T_2^{-1}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_2-ax_4\\x_3\\x_4\\x_5\end{pmatrix}.$$

Then

$$T_2^{-1}T_2\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=T_2^{-1}\begin{pmatrix}x_1\\x_2+ax_4\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_2+ax_4-ax_4\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}.$$

Therefore we have the following matrices

$$T_1=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}\text{ and }T_1^{-1}=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}$$

and we can easily confirm that

$$T_1T_1^{-1}=T_1^{-1}T_1=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}=I.$$

Similarly, we have

$$T_2=\begin{pmatrix}1&0&0&0&0\\0&1&0&a&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\text{ and }T_2^{-1}=\begin{pmatrix}1&0&0&0&0\\0&1&0&-a&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

and we can easily confirm that

$$T_2T_2^{-1} = T_2^{-1}T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a+a & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I.$$

6.12 a) We know the zero matrix, $\mathbf{0}$, is not invertible, therefore, letting

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we know that $A + B = \mathbf{0}$ is not invertible. But clearly A is invertible, since $A = I$ and clearly B is invertible since it can be easily checked that $BB = I$.

b) Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

then $A^2 = B^2 = \mathbf{0}$, and so it follows from problem 6.8 above that neither A nor B is invertible. On the other hand

$$A + B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = I = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Therefore $A + B$ is invertible and A and B are not invertible.

c) Let $A = B = I$, then

$$A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore, all of A, B and $A + B$ are invertible.

6.13 Suppose that A is invertible and $A = A^T$. In what follows we will show that A^{-1} is symmetric, that is $A^{-1} = (A^{-1})^T$. From the properties of inverses on page 27 of the text, we know that $(A^T)^{-1} = (A^{-1})^T$. Using this fact, together with the fact that $A = A^T$, we have

$$A^{-1} = (A^T)^{-1} = (A^{-1})^T.$$

Therefore, A^{-1} is symmetric.