Section 1.5 Solutions

5.1 a)

i. $AB \in M_{2\times 3}$
ii. BA is not defined.
iii. ABC is not defined.
iv. $ABD \in M_{2\times 1}$
v. BC is not defined.
vi. $BC^T \in M_{2\times 2}$
vii. $B^TC \in M_{3\times 3}$
viii. DC is not defined.
ix. $D^TC^T \in M_{1\times 2}$

b)

i. $AB = \begin{pmatrix} 7 & 2 & -2 \\ 6 & 1 & 4 \end{pmatrix}$
ii. $A(3B+C) = \begin{pmatrix} 18 & 6 & -5 \\ 19 & -2 & 20 \end{pmatrix}$
iii. $B^TA = \begin{pmatrix} 10 & 5 \\ 3 & 1 \\ -4 & 2 \end{pmatrix}$
iv. $A(BD) = \begin{pmatrix} -12 \\ -6 \end{pmatrix}$
v. $(AB)D = \begin{pmatrix} -12 \\ -6 \end{pmatrix}$

5.2 Recall that

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the identity matrix. Taking the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$, we have

$$T_\gamma(e_1) = (\cos(\gamma), \sin(\gamma))^T \text{ and } T_\gamma(e_2) = (-\sin(\gamma), \cos(\gamma))^T$$

and similarly,

$$T_{-\gamma}(e_1) = (\cos(\gamma), -\sin(\gamma))^T \text{ and } T_{-\gamma}(e_2) = (\sin(\gamma), \cos(\gamma))^T.$$

Therefore,

$$\begin{aligned} T_\gamma T_{-\gamma} &= \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \begin{pmatrix} \cos(\gamma) & \sin(\gamma) \\ -\sin(\gamma) & \cos(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\gamma) + \sin^2(\gamma) & \sin(\gamma)\cos(\gamma) - \sin(\gamma)\cos(\gamma) \\ \sin(\gamma)\cos(\gamma) - \sin(\gamma)\cos(\gamma) & \sin^2(\gamma) + \cos^2(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I, \end{aligned}$$

and one can similarly show that $T_{-\gamma}T = I$.

5.3 If T_α and T_β are rotation matrices, then

$$T_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

and

$$T_\alpha = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}.$$

Then, $T_\alpha T_\beta$ should correspond to rotation by $\alpha + \beta$, that is

$$T_\alpha T_\beta = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

On the other hand, using matrix multiplication, we know that

$$\begin{aligned} T_\alpha T_\beta &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta). \end{pmatrix} \end{aligned}$$

Combining these, we get that

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

5.5 Suppose that A and B are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the following associated matrices,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then we want

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which means that $ae = -bg$, $af = -bh$, $ce = -dg$. From this we can see that

$$a = \frac{-bg}{e} \text{ and } a = \frac{-bh}{f}$$

and combining these two facts

$$\begin{aligned} \frac{-bg}{e} &= \frac{-bh}{f} \\ \frac{g}{e} &= \frac{h}{f}. \end{aligned}$$

Therefore one possibility is that $e = f = -1$ and $g = h = 1$. Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -a+b & -a+b \\ -c+d & -c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So one possibility is that $a = b = -1$ and $c = d = -1$, then

$$AB = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

5.6 This problem follows the same setup as the example on page 20. Suppose that γ is the angle between the line $y = -\frac{2}{3}x$ and the x-axis. Let T_γ be the rotation by γ, which has the matrix

$$T_\gamma = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix},$$

let $T_{-\gamma}$ correspond to rotation by $-\gamma$, which has the matrix

$$T_{-\gamma} = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) \\ -\sin(\gamma) & \cos(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix},$$

and let R_0 be the linear transformation corresponding to reflection over the x-axis with matrix

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$T = T_\gamma R_0 T_{-\gamma} = \frac{1}{13} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}.$$

5.7 Let A be the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix},$$

and we want $a^2 + bc = 0$, $ab + bd = 0$, $ac + cd = 0$ and $bc + d^2 = 0$. From this we get that

$$0 = ab + bd = b(a + d)$$

and

$$0 = ac + cd = c(a + d)$$

so either let's suppose $a = -d$. Furthermore, we want $bc = -a^2 = -d^2$. So one possible solution is $a = 1$, $d = -1$, $b = 1$ and $c = -1$.