Section 1.4 Solutions

4.1 Vector space axiom 2 on page 1 of the textbook is the axiom of associativity. Recall that the sum of two linear transformations in $\mathcal{L}(V,W)$ is again a linear transformation (we proved this in class). Specifically, for $R, S \in \mathcal{L}(V,W)$, we define

$$\begin{aligned}(R+S) : V &\to W \\ \mathrm{v} &\mapsto R(\mathrm{v}) + S(\mathrm{v}).\end{aligned}$$

Recall also that two linear transformations are equal precisely when they send every element in their domain to the same image. Let $\mathrm{v} \in V$, and let $R, S, T \in \mathcal{L}(V,W)$, then

$$\begin{aligned}((R+S)+T)(\mathrm{v}) &= (R+S)(\mathrm{v}) + T(\mathrm{v}) \\ &= R(\mathrm{v}) + S(\mathrm{v}) + T(\mathrm{v}) \\ &= R(\mathrm{v}) + (S+T)(\mathrm{v}) \\ &= (R+(S+T))(\mathrm{v}).\end{aligned}$$

Therefore $(R+S)+T = R+(S+T)$.

4.2 Vector space axiom 8 on page 1 of the textbook is the distributive property. Recall that for any scalar α and for any $T \in \mathcal{L}(V,W)$, the map defined by

$$\begin{aligned}\alpha T : V &\to W \\ \mathrm{v} &\mapsto \alpha \cdot T(\mathrm{v})\end{aligned}$$

is a linear transformation. Recall also that two linear transformations are equal precisely when they send every element in their domain to the same image. Let α, β be scalars, let $T \in \mathcal{L}(V,W)$ and let $\mathrm{v} \in V$, then

$$(\alpha+\beta)T(\mathrm{v}) = (\alpha+\beta) \cdot T(\mathrm{v}) = \alpha \cdot T(\mathrm{v}) + \beta \cdot T(\mathrm{v}).$$

Therefore $(\alpha+\beta)T = \alpha T + \beta T$.