Section 1.3 Solutions

3.1 a) $\begin{pmatrix} 13 \\ 31 \end{pmatrix}$

b) $\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$

c) $\begin{pmatrix} 5 \\ 8 \\ 11 \\ 4 \end{pmatrix}$

d) These cannot be multiplied.

3.2 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3.3 a) $\begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$

c) If we consider that for any $0 \leq r \leq n$ we have $T(t^r) = rt^{r-1}$, we get the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & & 0 \\ 0 & 0 & 0 & 3 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

d) Let's examine the image of the basis elements under T,

$$\begin{aligned} T(1) &= 2 \cdot 1 \\ T(t) &= 2 \cdot t + 3 \cdot 1 \\ T(t^2) &= 2 \cdot t^2 + 3 \cdot 2 \cdot t - 4 \cdot 2 \\ &\vdots \\ T(T^n) &= 2 \cdot t^n + 3 \cdot nt^{n-1} - 4 \cdot n \cdot (n-1)t^{n-2} \end{aligned}$$

giving the matrix

$$\begin{pmatrix}
2 & 3 & -4(2) & 0 & \cdots & 0 & 0 \\
0 & 2 & 3(2) & 4(3)(2) & & 0 & 0 \\
0 & 0 & 2 & 3(3) & & 0 & 0 \\
0 & 0 & 0 & 2 & & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & & -4(n-1)(n-2) & 0 \\
0 & 0 & 0 & 0 & & 3(n-1) & -4(n)(n-1) \\
0 & 0 & 0 & 0 & & 2 & 3(n) \\
0 & 0 & 0 & 0 & \cdots & 0 & 2
\end{pmatrix}.$$

3.4 a) The transformation corresponding to projecting every vector of $\mathbb{R}^3$ onto the $x-y$ plane is

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \end{aligned}$$

Consequently, taking e_1, e_2, e_3 as the standard basis for $\mathbb{R}^3$, we have

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and hence the matrix corresponding to T is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) The transformation corresponding to reflecting every vector of $\mathbb{R}^3$ through the $x - y$ plane is

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} x \\ y \\ -z \end{pmatrix}. \end{aligned}$$

Consequently, taking e_1, e_2, e_3 as the standard basis for $\mathbb{R}^3$, we have

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

and hence the matrix corresponding to T is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

c) Let T be transformation corresponding to rotating the $x - y$ 30 degrees and leaving the z axis fixed. Taking e_1, e_2, e_3 as the standard basis for $\mathbb{R}^3$, we have

$$T(e_1) = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, T(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and hence the matrix corresponding to T is

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.5 Let $A : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Suppose that z is in the center of an interval $[x, y]$, this means

$$z = \frac{x+y}{2}.$$

Then, by the rules of linear transformations of vector spaces, we know that

$$Az = A(\frac{x+y}{2}) = \frac{1}{2}(A(x+y)) = \frac{Ax + Ay}{2}.$$

So by definition, Az is in the center of $[Ax, Ay]$.

3.6 a) Let $T : \mathbb{C} \to \mathbb{C}$ be the transformation in $\mathbb{C}$ corresponding to multiplication by $\alpha = a + ib$. For $x + iy$ and $u + iw$ in $\mathbb{C}$, we see that

$$\begin{aligned} T(x+iy+u+iw) &= T(x+u+i(y+w)) \\ &= (a+ib)(x+u+i(y+w)) \\ &= a(x+u)+ia(y+w)+ib(x+u)-b(y+w) \\ &= ax+iay+ibx-by+au+iaw+ibu-bw \\ &= (a+ib)(x+iy)+(a+ib)(u+iw) \\ &= T(x+iy)+T(u+iw) \end{aligned}$$

and for any scalar $c + id$ in $\mathbb{C}$ we have

$$\begin{aligned} T((c+id)(x+iy)) &= T(cx-dy+i(dx+cy)) \\ &= (a+ib)(cx+dy+i(dx+cy)) \\ &= (a+ib)(c+id)(x+iy) \\ &= (c+id)(a+ib)(x+iy) \\ &= (c+id)T(x+iy)), \end{aligned}$$

therefore T is a linear transformation. The basis for $\mathbb{C}$ is $\{1, i\}$, since every complex number is a linear combination of the form $x + iy$. Since

$$T(1) = a + bi, T(i) = -b + ia$$

we have the following matrix for T,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

b) Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, and identifying the complex number $\alpha = (a, b)^T$, we define

$$\begin{aligned} T : \mathbb{C} &\to \mathbb{C} \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \alpha \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Then we can show that T is a linear transformation, since for any $(x, y)^T, (u, w)^T \in \mathbb{C}$,

$$\begin{aligned} T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ w \end{pmatrix}\right) &= T\left(\begin{pmatrix} x+u \\ y+w \end{pmatrix}\right) = \alpha \begin{pmatrix} x+u \\ y+w \end{pmatrix} \\ &= \begin{pmatrix} \alpha(x+u) \\ \alpha(y+w) \end{pmatrix} \\ &= \begin{pmatrix} \alpha x + \alpha u \\ \alpha y + \alpha w \end{pmatrix} \\ &= \alpha \begin{pmatrix} x \\ y \end{pmatrix} + \alpha \begin{pmatrix} u \\ w \end{pmatrix} \\ &= T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + T\left(\begin{pmatrix} u \\ w \end{pmatrix}\right), \end{aligned}$$

and for any scalar $\beta \in \mathbb{C}$, we have

$$T\left(\beta \begin{pmatrix} x \\ y \end{pmatrix}\right) = T\left(\begin{pmatrix} \beta x \\ \beta y \end{pmatrix}\right) = \alpha \begin{pmatrix} \beta x \\ \beta y \end{pmatrix} = \beta\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \beta T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Now the basis for $\mathbb{C}$ when identified with $\mathbb{R}^2$ is $\{(1, 0)^T, (0, 1)^T\}$, and

$$T\left((1, 0)^T\right) = (\alpha, 0)^T, T\left((0, 1)^T\right) = (0, \alpha),$$

so T has the associated matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

c) Define $T(x + iy) = 2x - y + i(x - 3y)$. If we treat this as a linear transformation of the complex vector space $\mathbb{C}$, then recalling that $i^2 = -1$, we have

$$T(i^2) = iT(i) = i(-1 - 3i) = -i + 3$$

while on the other hand,

$$T(i^2) = T(-1) = 2 + i$$

which are not equal, violating the definition of a linear transformation. But treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, then T is just the linear transformation with associated matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}.$$

It still remains to be checked that in this case T satisfies the definition of a linear transformation, but this is very straightforward.

3.7 Suppose that $T : \mathbb{C} \to \mathbb{C}$ is a linear transformation. Then $T(1) = a+ib$ for some $a+ib \in \mathbb{C}$ and hence $T(-1) = -a - ib$. Since $i^2 = -1$, we have

$$-a - ib = T(-1) = T(i^2) = iT(i)$$

implying

$$T(i) = \frac{-a - ib}{i} = i(a + ib).$$

Therefore, for any $x + iy \in \mathbb{C}$,

$$T(x + iy) = T(x) + T(iy) = xT(1) + yT(i) = x(a + ib) + yi(a + ib) = (a + ib)(x + iy).$$