1 Complex Numbers

1.1 Sums and Products

**Definition:** The complex plane, denoted $\mathbb{C}$ is the set of all ordered pairs $(x, y)$ with $x, y \in \mathbb{R}$, where $\text{Re } z = x$ is called the *real part* and $\text{Im } z = y$ is called the *imaginary part*.

- We call the $x$-axis the *real axis*.
- We call the $y$-axis the *imaginary axis*.
- Usually we write $z = (x, y)$ or $z = x + iy$, this is called *rectangular form*.
- For two complex numbers, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ we have $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$

For $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, we define addition as

$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

and multiplication as

$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$

How do the real numbers interact here?

- For any $r \in \mathbb{R}$ we have $(r, 0) \in \mathbb{C}$, and

$(r, 0) + (s, 0) = (r + s, 0)$

and

$(r, 0)(s, 0) = (rs, 0)$

- For $i = (0, 1) \in \mathbb{C}$ we have

$i^2 = (0, 1)(0, 1) = (-1, 0)$

Now we can see that multiplication is well-defined, since

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i^2(y_1 y_2) + i(y_1 x_2 + x_1 y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$
1.2 Basic Algebraic Properties

- Complex numbers are **associative**
  \[ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \]
  and **commutative**
  \[ z_1 + z_2 = z_2 + z_1 \]
  so we can rearrange and drop parenthesis.

- We have an additive identity \( 0 = (0,0) \in \mathbb{C} \), and mult. identity \( 1 = (1,0) \in \mathbb{C} \). Check these!

- For any \( z = x + iy \) we have an **additive inverse** \(-z = -x - iy\) since
  \[ x + iy + (-x - iy) = 0. \]
  For \( z = x + iy \) non-zero we have a **multiplicative inverse** \( z^{-1} = u + iv \) so that \( zz^{-1} = 1 \) defined by
  \[(x + iy)(u + iv) = 1\]
  and so
  \[ux - vy + i(uy + vx) = 1\]
  therefore \( ux - vy = 1 \) and \( uy + vx = 0 \). Solving the linear equation, we get
  \[ z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \]

- If \( z_1 z_2 = 0 \) then \( z_1 = 0 \) or \( z_2 = 0 \). To see this, suppose \( z_1 \neq 0 \), then
  \[ z_2 = z_2 \cdot 1 = z_2 \cdot z_1^{-1} = 0 \cdot z_1^{-1} = 0 \]

- Some interesting things happen with quotients, for example
  \[ \frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i \]
  and
  \[ \frac{4 + i}{2 - 3i} = \frac{(4 + i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{5 + 14i}{13} = \frac{5}{13} + \frac{14}{13} \]
1.3 Vectors and Moduli

We can consider \( z = x + iy = (x, y) \) as a vector on the complex plane.

Then the sum can also be interpreted \textit{vectorially}, as follows.

\[
\begin{align*}
|z| &= \sqrt{x^2 + y^2} \\
|z|^2 &= (\text{Re}~z)^2 + (\text{Im}~z)^2
\end{align*}
\]

**Definition:** The \textit{modulus} of a complex number \( z = x + iy \) gives the distance from \( z \) to the origin, and is given by

\[
|z| = \sqrt{x^2 + y^2}
\]

extending the notion of absolute value for the reals, and

\[
|z|^2 = (\text{Re}~z)^2 + (\text{Im}~z)^2
\]

now gives a relation between real numbers. From here

- \(|z| = |z| = |z|
- \( \text{Re}~z \leq |z| \) and \(|z| \leq |z| \)
- \( \text{Im}~z \leq |z| \) and \(|z| \leq |z| \)
- \( \text{Note that } z_1 < z_2 \text{ is meaningless, but } |z_1| < |z_2| \text{ has meaning.} \)

Consider the difference

\[
|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}
\]

What does this mean geometrically?
Examples:

1. Consider
   \[ | -3 + 2i | = \sqrt{9 + 4} = \sqrt{13} \]
   and
   \[ | 1 + 4i | = \sqrt{1 + 16} = \sqrt{17} \]
   What does this mean, geometrically?

2. The equation \( |z| = 1 \) is the unit circle centered at the origin.

3. The equation \( |z - 1 + 3i| = 2 \) is the circle with center \( z = (1, -3) \) and radius 2.

4. Exercise: Show that \( |z_1 \cdot z_2| = |z_1| \cdot |z_2| \).

5. Exercise: Show that \( |z^2| = |z|^2 \).

1.4 Triangle Inequality

**Definition:** The *triangle inequality* is given by

\[ |z_1 + z_2| \leq |z_1| + |z_2| \]

Illustrated by figure 3. Some consequences:

- We have \( |z_1 + z_2| \geq |z_1| - |z_2| \), to see this, observe
  \[ |z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2| \]
  and hence
  \[ |z_1 + z_2| \geq |z_1| - |z_2| \text{ if } |z_1| \geq |z_2| \]
  and
  \[ |(z_1 + z_2)| \geq -(|z_1| - |z_2|) \text{ if } |z_1| < |z_2| \]

What does this mean geometrically? Length of one side is greater than difference of other two.
Examples:

1. Suppose $z$ is on the circle $|z| = 1$. Then
   
   $$|z - 2| = |z + (-2)| \leq |z| + |2| = 1 + 2 = 3$$

   and
   
   $$|z - 2| = |z + (-2)| \geq ||z| - | -2|| = |1 - 2| = 1.$$

   This means the distance from $z$ to 2 is between 1 and 3.

2. The triangle inequality can be extended inductively, that is
   
   $$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

   for any $n$. So for example, for $z$ on the circle $|z| = 2$, we have
   
   $$|3 + z + z^2| \leq 3 + |z| + |z^2| = 3 + |z| + |z|^2 \leq 9.$$

3. Consider the degree $n$ polynomial given by
   
   $$P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$$

   where $n \in \mathbb{Z}^+$ and $a_0, \ldots, a_n \in \mathbb{C}$. We will show that for some positive $R \in \mathbb{R}$,
   
   $$\left| \frac{1}{P(z)} \right| < \frac{2}{a_nR^n}$$

   whenever $|z| > R$. Morally: Reciprocal $1/P(z)$ is bounded from above when $z$ is outside of the circle $|z| = R$. Recall,
   
   • $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
   • $|z^2| = |2|^2$

   and write
   
   $$w = \frac{P(z) - a_nz^n}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z}.$$ 

   Then
   
   $$P(z) = (a_n + w)z^n$$

   Then the equation above becomes
   
   $$wz^n = a_0 + a_1z + \ldots + a_{n-1}z^{n-1}$$

   and hence
   
   $$|w||z^n| \leq |a_0| + |a_1|z + \ldots + |a_{n-1}|z^{n-1}$$

   implying
   
   $$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|}.$$
Now pick $R$ large enough so that
\[ \frac{|a_i|}{|z|^{n-1}} < \frac{|a_n|}{2n} \]
when $|z| > R$. Then,
\[ w < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2} \]
whenever $|z| < R$.

Consequently,
\[ |a_n + w| \geq |a_n| - |w| > \frac{|a_n|}{2} \]
and hence
\[ |P(z)| = |a_n + w| \cdot |z^n| > \frac{|a_n|}{2} \cdot R^n. \]

1.5 Complex Conjugates

**Definition:** The complex conjugate of a number $z = x + iy$ is given by
\[ \bar{z} = x - iy \]

Some observations:
- $|z| = |ar{z}|$
- $\bar{\bar{z}} = z$
- $\bar{z_1 + z_2} = \bar{z_1} + \bar{z_2}$, since
\[ \bar{z_1 + z_2} = x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \bar{z_1} + \bar{z_2}. \]
- $\bar{z_1 - z_2} = \bar{z_1} - \bar{z_2}$
- $\bar{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$
- $\frac{\bar{z_1}}{\bar{z_2}} = \frac{\bar{z_1}}{\bar{z_2}}$
- $z + \bar{z} = 2\text{Re } z$
- $z - \bar{z} = 2i\text{Im } z$
- $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$

Examples:
1. $\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{5} = -1 + i$
2. We can use conjugates to establish modulus properties. We know \( z \overline{z} = |z|^2 \). Therefore, we obtain

\[
|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1} \cdot \overline{z_2}) = z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2} = |z_1|^2 \cdot |z_2|^2 = (|z_1| \cdot |z_2|)^2
\]

from which we conclude that \( |z_1 \cdot z_2| = |z_1| \cdot |z_2| \), since modulus is always positive.

1.6 Exponential Form

Let \( r > 0 \) and \( \theta \) be polar coordinates for \( z = (x, y) \), then

\[
z = r(\cos \theta + i \sin \theta)
\]

then

\[
|z| = r \sqrt{\cos^2 \theta + \sin^2 \theta} = r
\]

**Definition:** The value \( \theta = \text{arg}(z) \) is an *argument* for \( z \), and it is the angle \( z = (x, y) \) makes with the real axis. Infinite arguments exist for \( z \), namely,

\[
\theta + 2\pi, \theta + 4\pi, ..., \theta + 2k\pi, ...
\]

but we call \( \Theta = \text{Arg}(z) \) with \( -\pi < \Theta \leq \pi \) the *principal argument* of \( z \).

Examples:

1. The complex number \(-1 - i\) has

\[
\text{Arg}(-1 - i) = \frac{-3\pi}{4}
\]

but

\[
\text{arg}(-1 - i) = \frac{5\pi}{4} + 2k\pi
\]

for \( n \in \mathbb{Z} \).
Definition: The symbol $e^{i\theta}$ is defined by Euler’s formula as

$$e^{i\theta} = \cos \theta + i \sin \theta$$

and hence

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

Examples:

1. Writing $-1 - i$ in exponential form we have

$$-1 - i = \sqrt{2} e^{i \frac{-3\pi}{4}}$$

or

$$-1 - i = \sqrt{2} e^{i \left( \frac{-3\pi}{4} + 2\pi k \right)}$$

2. The set of numbers $z = e^{i\theta}$ are the set of numbers on the unit circle. Accordingly,

$$e^{i\pi} = -1$$

and $e^{i\pi/2} = i$ and $e^{i4\pi} = 1$.

Also,

$$z = Re^{i\theta} \text{ where } 0 \leq \theta \leq 2\pi$$

is a parametrization of the unit circle, moving counter-clockwise, and

$$z = z_0 + Re^{i\theta} \text{ where } 0 \leq \theta \leq 2\pi$$

moves the circle to be centered at $z_0$.

Some basic properties of exponential forms:

- $e^{i\theta} e^{i\theta'} = e^{i(\theta + \theta')}$
- $re^{i\theta} \cdot r'e^{i\theta'} = (rr') \cdot e^{i(\theta + \theta')}$
- $\frac{re^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'} e^{i(\theta - \theta')}$
- If $z = re^{i\theta}$ then

$$z^{-1} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

and

$$z^n = r^n e^{in\theta}.$$

Examples:
1. From the properties above we see that

$$\text{arg}(z_1z_2) = \text{arg}(z_1) + \text{arg}(z_2).$$

but this doesn’t necessarily hold when arg is replaced with Arg. For example, $z_1 = -1$ and $z_2 = i$, then

$$\text{Arg}(-1) = \pi \text{ and } \text{Arg}(i) = \frac{\pi}{2}$$

so

$$\text{Arg}(-1) + \text{Arg}(i) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

but

$$\text{Arg}(-1 \cdot i) = \text{Arg}(-i) = -\frac{\pi}{2}$$

2. Find the principal argument of $\frac{i}{-1 - i}$.

We already know that $i = e^{i\frac{\pi}{2}}$ and $-1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$ and so

$$\frac{i}{-1 - i} = \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}e^{-i\frac{3\pi}{4}}} = \frac{1}{\sqrt{2}}e^{i(\pi/2 + 3\pi/4)}$$

Therefore

$$\text{arg} \left( \frac{i}{-1 - i} \right) = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4}$$

and hence

$$\text{Arg} \left( \frac{i}{-1 - i} \right) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

3. Let’s write the expression $(-1 + i)^7$ in rectangular form. First, for $z = (-1 + i)$ we have

$$|z| = \sqrt{1 + i} = \sqrt{2}$$

and

$$\text{Arg}(z) = \frac{3\pi}{4}$$

so $-1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$, and

$$(-1 + i)^7 = \left( \sqrt{2}e^{i\frac{3\pi}{4}} \right)^7 = 8\sqrt{2}e^{i\frac{21\pi}{4}} = 8\sqrt{2}e^{i5\pi/4}e^{i\pi} = -8\sqrt{2}e^{i\pi}$$

But we can easily see that

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

and so

$$(-1 + i)^7 = -8(1 + i)$$
4. When $r = 1$, for any value $n \in \mathbb{Z}$, we have
\[
(e^{i\theta})^n = e^{in\theta}
\]
but this means
\[
(cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta
\]
this is known as De Moivre’s formula.

5. Using the above with $n = 2$, we reclaim known identities
\[
\cos 2\theta - \sin^2 \theta + 2i \cos \theta \sin \theta = (\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta.
\]
Now equating real and imaginary parts, we have
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin 2\theta = 2 \cos \theta \sin \theta.
\]

1.7 Roots of Complex Numbers

Two non-zero complex numbers can be arrived at in infinitely many ways, for example
\[
z = re^{i\theta} \quad \text{and} \quad z' = r'e^{i\theta'}
\]
are equal precisely when
\[
r = r' \quad \text{and} \quad \theta = \theta' + 2\pi k
\]
for $k \in \mathbb{Z}$. We also know that $z^n = r^n e^{i\theta}$, so we should know something about $n^{th}$ roots.

**Definition:** An $n^{th}$ root of $z_0 \in \mathbb{C}$ is a complex number $z = re^{i\theta}$ such that
\[
z^n = z_0
\]
that is
\[
r_0 e^{i\theta_0} = r^n e^{in\theta},
\]
and hence
\[
r_0 = r^n \quad \text{and} \quad n\theta = \theta_0 + 2\pi k.
\]

- Since $r_0$ is real, this just means $r = \sqrt[n]{r_0}$.
- For $k \in \mathbb{Z}$, we know that
\[
\theta = \theta_0 + \frac{2\pi k}{n}
\]
So the $n^{th}$ roots of $z_0$ are given by the infinite family
\[
z = \sqrt[n]{r_0} \exp \left[ i \left( \frac{\theta_0}{n} + \frac{2\pi k}{n} \right) \right]
\]
all lying on the circle $|z| = \sqrt[n]{r_0}$.

So $n$ of them should be more special than the others, namely, those with
\[
c_k = \sqrt[n]{r_0} \exp \left[ i \left( \frac{\theta_0}{n} + \frac{2\pi k}{n} \right) \right]
\]
and $k \in 0, 1, 2, \ldots, n - 1$. 

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• When \( z_0 \) is a positive real then \( \sqrt[n]{r_0} \) means the familiar unique positive root.
• When \( \theta_0 = \text{Arg}(z_0) \) then we call \( c_0 \) the principal root.
• When \( z_0 \) is a positive real, its principal root is \( \sqrt[n]{r_0} \).

Examples:
1. Let’s find all values of \((-16)^{\frac{1}{4}}\), that is, the 4th roots of \(-16\). Since

\[
-16 = 16e^{i(\pi + 2k\pi)}
\]

so the roots are just

\[
c_k = 2e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)}
\]

where \( k = 0, 1, 2, 3 \). So we have

\[
c_k = 2e^{i\left(\frac{\pi}{4}\right)}e^{i\left(\frac{k\pi}{2}\right)} = 2 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) e^{i\left(\frac{k\pi}{2}\right)} = \sqrt{2}(1 + i)e^{i\left(\frac{k\pi}{2}\right)}
\]

and therefore

\[
c_0 = \sqrt{2}(1 + i) \cdot 1 = \sqrt{2}(1 + i)
\]
\[
c_1 = \sqrt{2}(1 + i) \cdot i = \sqrt{2}(-1 + i)
\]
\[
c_2 = \sqrt{2}(1 + i) \cdot -1 = \sqrt{2}(-1 - i)
\]
\[
c_3 = \sqrt{2}(1 + i) \cdot -i = \sqrt{2}(1 - i)
\]

2. Let’s examine the \( n \)th roots of 1, called the \( n \)th roots of unity. We start with

\[
1 = 1 \cdot e^{i(0 + 2\pi k)}
\]

where \( k \in \mathbb{Z} \). Then,

\[
c_k = \sqrt[n]{1} \cdot e^{i\left(0 + \frac{2\pi k}{n}\right)} = e^{i\left(\frac{2\pi k}{n}\right)}
\]

where \( k \in \{0, 1, 2, \ldots, n - 1\} \). When \( n = 2 \) we get \( c_0 = 1 \) and \( c_1 = -1 \). For larger \( n \), we get roots lying on the regular \( n \)-gon inscribed in the circle \(| z | = 1\).
1.8 Regions in the Complex Plane

**Definition:** We define an $\epsilon$ **neighborhood** around a given point $z_0$ by

$$|z - z_0| < \epsilon$$

we can also define the **deleted neighborhood**, that is the neighborhood minus the point $z_0$ itself, by

$$0 < |z - z_0| < \epsilon$$

For a given set $S$ contained in $\mathbb{C}$,

- A point $z_0$ is called **interior** if there exists a neighborhood containing only $z_0$ and points of $S$.
- A point $z_0$ is called **exterior** if there exists a neighborhood containing $z_0$ and no points in $S$.
- If $z_0$ is neither of these, then it is a **boundary point**.
- A set is **open** if it contains none of its boundary points.
- A set is **closed** if it contains all of its boundary points.
- A set is a **closure** of $S$ if it contains $S$ plus its boundary points. For example, $|z| \leq 1$ is the closure of $|z| < 1$.
- Some sets like $0 < |z| \leq 1$ is neither open nor closed.
• An open set $S$ is \textit{connected} if every pair of points in $S$ can be joined by a polygonal line.
• A \textit{domain} is an non-empty open connected set.
• A domain with some, none, or all of its boundary points is called a region.
• A set $S$ is \textit{bounded} if every point of $S$ lies in some circle $|z| < R$.

\textbf{Examples:}

1. Consider the set $\text{Im} \left( \frac{1}{z} \right) > 1$ in relation to these properties, for $z \neq 0$ we have

\[
\frac{1}{z} = \frac{\overline{z}}{zz} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}
\]

so we want

\[
\text{Im} \left( \frac{1}{z} \right) = \frac{-y}{x^2 + y^2} > 1
\]

and therefore

\[
x^2 + y^2 + y < 0.
\]

Completing the square, we get

\[
x^2 + y^2 + y + \frac{1}{4} < \frac{1}{4}
\]

which means

\[
(x - 0)^2 + \left( y + \frac{1}{2} \right)^2 < \left( \frac{1}{2} \right)^2.
\]

This is the region whose boundary is the circle centered at $x = 0$ and $y = -1/2$ with radius $1/2$.

• A point $z_0$ is an \textit{accumulation point}, or limit point, of a set $S$ if each deleted neighborhood of $z_0$ contains at least one point of $S$. 