Theorem. The real number $\sqrt{2}$ is irrational.

Proof. To prove that $\sqrt{2}$ is irrational, we proceed by contradiction. In particular, we assume that $\sqrt{2}$ is rational. This tells us that there exist integers m and n with $n \neq 0$ such that

$$\sqrt{2} = \frac{m}{n}.$$

We further assume that m and n have no common factors so that $\frac{m}{n}$ is in lowest terms. Let us now consider the consequences of assuming that $\sqrt{2} = \frac{m}{n}$. Our goal is to show that this cannot happen.

Squaring both sides of $\sqrt{2} = \frac{m}{n}$, we find that $2 = \frac{m^2}{n^2}$, so that multiplying by n^2, we have

$$m^2 = 2n^2. \tag{1}$$

Since n^2 is an integer, it follows that m^2 is even. By Theorem 3.10, which states that an integer is even if and only if its square is even, we now know that since m^2 is even, it follows that m is also even. In particular, there exists some integer k such that

$$m = 2k. \tag{2}$$

If we now substitute $2k$ for m in Equation (1), it follows that $(2k)^2 = 2n^2$. Dividing both sides by 2, we find that

$$n^2 = 2k^2. \tag{3}$$

Now Equation (3) implies that n^2 is even, and therefore by again applying Theorem 3.10, n must also be even. In particular, there exists some integer p such that

$$n = 2p. \tag{4}$$

Now consider Equations (2) and (4). These show that $2 \mid m$ and $2 \mid n$, so that m and n share a common factor of 2, and thus $\frac{m}{n}$ is not in lowest terms. This is a contradiction. Moreover, this contradiction proves that our assumption that $\sqrt{2}$ is rational is false, and therefore $\sqrt{2}$ is irrational. $\square$