

Theorem. Let A and B be subsets of a universal set U . Then, **the following conditions are equivalent:**

- (1.) $A \subseteq B$;
- (2.) $A \cap B^c = \phi$;
- (3.) $A^c \cup B = U$.

Proof. We will prove that the above three statements are equivalent by showing the following three implications: (1) \Rightarrow (2), (2) \Rightarrow (3), and (3) \Rightarrow (1).

To begin, we show that (1) \Rightarrow (2). In particular, we must show that for all sets A and B in U , if $A \subseteq B$, then $A \cap B^c = \phi$. We will use a contrapositive argument to prove this, and hence assume that A and B are sets such that $A \cap B^c \neq \phi$. We will prove that $A \not\subseteq B$.

Since $A \cap B^c \neq \phi$, there exists some $y \in A \cap B^c$. Because $y \in A \cap B^c$, by definition we know that $y \in A$ and $y \in B^c$. The latter tells us that $y \notin B$. Thus we have shown that there exists an element y such that $y \in A$ but $y \notin B$, and therefore by definition, $A \not\subseteq B$. Hence (1) \Rightarrow (2).

Next we will prove directly that (2) \Rightarrow (3) using the algebra of set operations. We are to prove that for all sets A and B , if

$$A \cap B^c = \phi, \tag{1}$$

then

$$A^c \cup B = U. \tag{2}$$

We thus assume that Equation (1) is true, and will show that Equation (2) follows. Taking the complement of both sides of Equation (1), we have

$$(A \cap B^c)^c = (\phi)^c.$$

By DeMorgan's Law applied to the lefthand side, together with our knowledge that the complement of the empty set is the universal set, it follows that

$$A^c \cup (B^c)^c = U.$$

Because $(B^c)^c = B$, we see that we have proven that Equation (2) indeed holds, and thus (2) \Rightarrow (3).

Finally, we must prove that (3) \Rightarrow (1). In particular, we must show that for any sets A and B , if $A^c \cup B = U$, then $A \subseteq B$. We will do this directly using the Choose Method. First, we assume that A and B are sets in U and that $A^c \cup B = U$. To prove that $A \subseteq B$, we must show that for every $x \in U$, if $x \in A$,

then $x \in B$. Thus, we choose $x \in A$. Observe that this implies that $x \notin A^c$. Now, since we also know that $A^c \cup B = U$, every element in the universal set must be in at least one of A^c or B . Since our particular element x is not in A^c , it follows that $x \in B$. This proves that $A \subseteq B$, and therefore $(3) \Rightarrow (1)$.

Having proven that $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, and $(3) \Rightarrow (1)$, it follows that these three statements are indeed equivalent and the theorem is true. \square

You can compare a different version of this proof by reading Theorem 5.22 in your text. It is also possible to prove $(1) \Rightarrow (2)$ by a contradiction argument.