PROOF PORTFOLIO: PROBLEM GROUP B

Directions. For each conjecture in the portfolio, determine whether the statement is true or false. If the statement is true, prove it. If not, provide a valid counterexample, as well as a slightly modified statement that is true, and prove the slightly modified statement.

If a biconditional statement is found to be false, you should clearly determine if one of the conditional statements within it is true; in that case, you should provide a proof of the conditional statement that holds. A counterexample for the statement that fails to hold suffices in this case (that is, you need not include an updated true statement and its proof for the one direction that is false).

Problems 4-7

Choose 4 of the following 8 problems to complete Group B, according to these guidelines: choose one of B1 and B2, one of B3-B5, and two of B6-B8.

Conjecture B1. The real number $\log_2 3$ is irrational.

Conjecture B2.[1] The real number $\sqrt{14} - \sqrt{42}$ is irrational.

Problem B3. Conjecture and prove a formula for the nth derivative of $f(x) = x^2 e^x$.

Conjecture B4. For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{4i^2 - 1} = \frac{n}{2n + 1}$$

Conjecture B5.[2] For every natural number n,

$$(2!)(4!)(6!) \cdots ((2n)!) \geq ((n+1)!)^n.$$

Conjecture B6. If S and T are the sets defined by $S = \{x \in \mathbb{Z} \mid x \equiv 5 \pmod 7\}$ and $T = \{x \in \mathbb{Z} \mid x^2 \equiv 4 \pmod 7\}$, then T is a proper subset of S.

Conjecture B7. For any sets X and Y in a universal set U, $(X \cap Y^c) \cup (Y \cap X^c) = X$ if and only $Y = \emptyset$.

Conjecture B8. Let X and Y be sets and f be a function such that $f : X \to Y$. If $A \subseteq X$ and $B \subseteq X$, then $f(A) \backslash f(B) \subseteq f(A \backslash B)$. (Note: $f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}$.)

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[1]You may use theorems about square roots of prime numbers after we discuss them in class.
[2]Hint: At some point it may be helpful to use the fact that $(2n + 2)! = (2n + 2)(2n + 1) \cdots (n + 3)(n + 2)!$, as well to remember that multiplication is both commutative and associative.