6.3: INJECTIONS, SURJECTIONS, AND BIJECTIONS SAMPLE PROOF

Theorem. The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 2x^3 - 5$ is both an injection and a surjection.

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 2x^3 - 5$.

We will first prove that g is an injection. By definition, this means we must show that for all $a_1, a_2 \in \mathbb{R}$, if $g(a_1) = g(a_2)$, then $a_1 = a_2$. Thus, we assume that a_1 and a_2 are real numbers such that $g(a_1) = g(a_2)$. From this, it follows that

$$2a_1^3 - 5 = 2a_2^3 - 5.$$

Adding 5 to both sides and then dividing by 2, we find that

$$a_1^3 = a_2^3.$$

Because the function $f(x) = x^3$ is itself known to be an injective function, the last equation implies that $a_1 = a_2$, and thus we have shown that g is an injection.

To prove that g is also a surjection, we must prove that for each real number b, there exists a real number a such that $g(a) = b$. We do this by a constructive proof. Choose $b \in \mathbb{R}$ and let a be given by the formula

$$a = \sqrt[3]{\frac{b+5}{2}}.$$

Since b, 5, and 2 are real numbers and 2 is nonzero, it follows that $\frac{b+5}{2}$ is also a real number, by the closure of the real numbers under addition and nonzero division. Furthermore, since the cube root of any real number is another real number, it is also the case that $a \in \mathbb{R}$.

Now, consider $g(a)$. Observe that by the rule for g and standard algebra, it follows that

$$\begin{aligned} g(a) &= g\left(\sqrt[3]{\frac{b+5}{2}}\right) \\ &= 2\left(\sqrt[3]{\frac{b+5}{2}}\right)^3 - 5 \\ &= 2\left(\frac{b+5}{2}\right) - 5 \\ &= (b+5) - 5 \\ &= b. \end{aligned}$$

Thus, given an arbitrary real number b, we have constructed a real number a such that $g(a) = b$, which proves by definition that g is a surjection. □